Towards the exact solutions of (super)membrane models

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Plan

1. Membranes
- $U(1)$ invariant Membranes

(joint work with J. Hoppe and A. Zheltukhin, in preparation)

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2. Supermembranes
- the ground state around the origin

(joint work with J. Hoppe and D. Lundholm, arXiv:0809.5270)

Membranes

- classically - `Dirac 62'`

$$S_{NG} = \int_\Sigma d^3\sigma \sqrt{|G|}, \quad G_{ab} = \partial_a X^\mu \partial_b X_\mu,$$

$$\partial_a(\sqrt{|G|} G^{ab} \partial_b X^\mu) = 0,$$

$$\sigma^a = (\tau, \sigma^1, \sigma^2), \quad \mu = 0, \dots, D-1 = 4$$

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- EOM become ($g_{rs} := -G_{rs}$, induced metric on Σ)

$$\ddot{\vec{X}} = \frac{1}{\rho} \partial_r \left(\frac{g}{\rho} g^{rs} \partial_s \vec{X} \right), \quad \dot{\rho} = 0, \quad \rho := \sqrt{\frac{g}{1 - \dot{\vec{X}}^2}}$$

- U(1) symmetric ansatz, $\rho = 1$
  $\vec{X}^T = (m_1 \cos\sigma_2, m_1 \sin\sigma_2, m_2 \cos\sigma_2, m_2 \sin\sigma_2)$

$$\text{EOM} \rightarrow \qquad \ddot{\vec{m}} = (\vec{m}^2 \vec{m}')' - \vec{m}'^2 \vec{m} = 0,$$

$$C \rightarrow \qquad \dot{\vec{m}}^2 = \vec{m}^2 \vec{m}' - 1 = 0, \qquad \dot{\vec{m}} \vec{m}' = 0, \qquad (C \Rightarrow EOM!)$$

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- **Zero curvature representation**
  in $\xi^\alpha = (\tau, \sigma^1 = \sigma)$ space introduce a frame (*repère*) $\vec{n}_i$

$$\vec{n}_0 = \frac{\dot{\vec{m}}}{\sqrt{1 - \vec{m}^2 \vec{m}'^2}}, \qquad \vec{n}_1 = \frac{\vec{m}'}{\sqrt{\vec{m}'^2}}, \qquad \vec{n}_i \cdot \vec{n}_j = \delta_{ij}$$

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- define
  $$d\vec{m} = \omega_i \vec{n}_i, \quad d\vec{n}_j = \Gamma^k_i \vec{n}_k \qquad (\partial_\alpha \vec{m} = \omega^i_\alpha \vec{n}_i, \;\; \partial_\alpha \vec{n}_i = \Gamma^k_{i\alpha} \vec{n}_k)$$

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- M-C equations $d^2 \vec{m} = 0$ give

$$\partial_{[\alpha} \omega^i_{\beta]} + \Gamma^i_{k[\alpha} \omega^k_{\beta]} = 0$$

- using $\Gamma^i_{k\alpha} =: \epsilon^i_k \mathcal{A}_\alpha$ the M-C equations become

$$\partial_\sigma \sqrt{1 - \vec{m}^2 \vec{m}'^2} - \mathcal{A}_\tau \sqrt{\vec{m}'^2} = 0$$

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$$\frac{A'}{B} = \dot{\phi}, \qquad \frac{\dot{B}}{A} = -\phi'$$

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- Riemann method

Supermembranes

- quantum description → $SU(N)$ matrix regularization:

$$H_{reg.} = Tr\left(\frac{1}{2}P_iP_i - \frac{1}{4}[X_i, X_j][X_i, X_j] - \frac{i}{2}X^i\theta\gamma^i\theta\right)$$

$$X_i = x_{iA}T_A, \quad P_i = p_{iA}T_A \quad \theta_\alpha = \theta_{\alpha A}T_A, \quad T_A \in su(N)$$

$$[x_{jA}, p_{kB}] = i\delta_{jk}\delta_{AB}, \quad \{\theta_{\alpha A}, \theta_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}, \quad \gamma^i - 16\times 16, real$$

-Goldstone, Hoppe 82', de Wit, Hoppe, Nicolai, 87'

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- Other applications
  - femotouniverse of YM → YMQM - Bjorken 79'
  - small volume YM - Lüscher 82'
  - eff. description of D0 branes → YMQM - Leigh 89'
  supersymmetric case → SYMQM - Witten 96'
  - M theory on a light cone in IMF → SYMQM
  - Banks, Fischler, Shenker, Susskind 97'

- **does the ground state exist?** $Q_\alpha \Psi = 0$

$$H = Q_\alpha^2/2, \qquad Q_\beta = (-i\gamma^j_{\beta\alpha}\partial_{jA} + \frac{1}{2}\gamma^{jk}_{\beta\alpha} f_{ABC} x_{jB} x_{kC})\theta_{\alpha A}$$

- the Witten index computations suggest: YES

`Yi 97', Sethi, Stern 98', Moore, Nekrasov, Shatashvili 98'`

- recent progress (deformation/averaging techniques)

`J. Hoppe, D. Lundholm, M.T, arXiv:0803.1316, 0809.5271`

$\rightarrow$ **see D. Lundholm's poster!**

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- **can it be constructed?**

  - asymptotic behavior - `Bach, Bordemann, Fröhlich, Graf, Halpern, Hasler, Hoppe, Konechny, Lundholm, Plefka, Schwatrz, Suter, Yau - 97'-07'`

  - near the origin ...

- consider the Taylor expansion

$$\Psi(x,\theta) = \psi^{(0)} + x_{iA}\psi^{(1)}_{iA} + \frac{1}{2}x_{iA}x_{jB}\psi^{(2)}_{iA\,jB} + \dots$$

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- $Q_\beta \Psi = 0$ gives 3 towers of equatins (for $\psi^{(3k)}$, $\psi^{(3k+1)}$ and $\psi^{(3k+2)}$)
- no constraints on $\psi^{(0)}$?
  $Spin(9) \times SU(2)$ invariance $\rightarrow \epsilon_{ABC}\theta_{\alpha A}\theta_{\alpha B}\psi^{(0)} = 0$

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- $\psi^{(0)}$ could be some combination of e.g.

$$|ik\rangle_1 |jk\rangle_2 |ij\rangle_3, \quad |i\alpha\rangle_1 |j\alpha\rangle_2 |ij\rangle_3, \quad |ikl\rangle_1 |jkl\rangle_2 |ij\rangle_3$$

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  $$\rightarrow |ij\rangle,\ |ijk\rangle,\ |i\alpha\rangle$$
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- we need $\theta_{\alpha A}|ij\rangle_A = ?$, $\quad \theta_{\alpha A}|ijk\rangle_A = ?$, $\quad \theta_{\alpha A}|i\alpha\rangle_A = ?$

- It turns out that for

$$\underset{44}{|||}\,1\rangle := |ik\rangle_1|jk\rangle_2|ij\rangle_3,$$

$$\underset{844}{|||}\,1\rangle := |ikl\rangle_1|jkl\rangle_2|ij\rangle_3 + |jkl\rangle_1|ij\rangle_2|ikl\rangle_3 + |ij\rangle_1|ikl\rangle_2|jkl\rangle_3$$

the combination

$$\phi := \underset{44}{|||}\,1\rangle + \frac{13}{36}\underset{844}{|||}\,1\rangle$$

is the (unique) $Spin(9) \times SU(2)$ invariant state, $\psi^{(0)} \sim \phi$ (no **128** content)