Bilinear discretization of quadratic vector fields: an overview of recent results

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The Problem of integrable discretization

Consider a completely integrable flow on $(P, \{\cdot,\cdot\})$:

$$\dot{x} = f(x) = \{H, x\} \qquad \exists \{I_k(x)\} \in \mathcal{F}(P).$$

Problem of integrable discretization

To find a family of diffeomorphisms $\phi_\epsilon : P \to P$,

$$\tilde{x} = \phi_\epsilon(x) \qquad 0 < \epsilon \ll 1,$$

- $\phi_\epsilon(x) = x + \epsilon\, f(x) + O(\epsilon^2)$.
- The maps are Poisson w.r.t. $\{\cdot,\cdot\}$ on P or w.r.t. some its deformation, $\{\cdot,\cdot\}_\epsilon = \{\cdot,\cdot\} + O(\epsilon)$.
- The maps possess the necessary number of independent integrals in involution $I_k(x,\epsilon) = I_k(x) + O(\epsilon)$.

Hirota-Kimura discretization of the Euler top

$$\dot{x}_i = \alpha_i x_j x_k,$$

$x = (x_1, x_2, x_3) \in \mathbb{R}^3$ vector of coordinates, $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ vector of parameters, (ijk) cyclic permutation of (123).

Integrability features:

1. Bi-Hamiltonian structure.
2. Two independent integrals of motion (in involution w.r.t. both PBs).
3. Lax representation.
4. Explicit solutions in terms of elliptic functions.

- A class of implicit discretizations [BLS][1]:

$$\tilde{x}_i - x_i = \alpha_i \gamma(x, \epsilon)(\tilde{x}_j + x_j)(\tilde{x}_k + x_k),$$

  where tilde denotes the shift $t \mapsto t + \epsilon$ in $\epsilon\mathbb{Z}$.

- Hirota-Kimura discretization [HK][2]:

$$\tilde{x}_i - x_i = \epsilon \alpha_i (\tilde{x}_j x_k + x_j \tilde{x}_k).$$

[1] A.I. Bobenko, B. Lorbeer, Yu.B. Suris, *Jour. Math. Phys.*, 1998, **39**

[2] R. Hirota, K. Kimura, *Jour. Phys. Soc. Japan*, 2000, **69**

Hirota-Kimura discretization of the Euler top

The map $\tilde{x}_i - x_i = \epsilon\alpha_i(\tilde{x}_j x_k + x_j \tilde{x}_k)$ can be written as

$$\tilde{x} = f(x,\epsilon) = A^{-1}(x,\epsilon)x, \qquad A(x,\epsilon) = \begin{pmatrix} 1 & -\epsilon\alpha_1 x_3 & -\epsilon\alpha_1 x_2 \\ -\epsilon\alpha_2 x_3 & 1 & -\epsilon\alpha_2 x_1 \\ -\epsilon\alpha_3 x_2 & -\epsilon\alpha_3 x_1 & 1 \end{pmatrix}.$$

1. Equations are linear w.r.t. $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$. They give an explicit rational map.
2. The map is reversible (therefore birational): $f^{-1}(x,\epsilon) = f(x,-\epsilon)$.
3. Two independent integrals of motion:

$$F_i = \frac{1-\epsilon^2\alpha_k\alpha_i x_j^2}{1-\epsilon^2\alpha_i\alpha_j x_k^2}, \qquad F_1F_2F_3 = 1.$$

4. Explicit solutions in terms of elliptic functions.
5. Invariant measure form and bi-Hamiltonian structure [PS][3].
6. A Lax representation is still missing.

[3] M. Petrera, Yu.B. Suris, to appear in *Math. Nach.*, 2008

Hirota-Kimura or Kahan?

- The Hirota-Kimura discretization of the Euler top seemed to be an isolated curiosity.
- Kahan discretization of quadratic vector fields [K][4]:

$$\dot{x} = Q(x) + Bx,$$

$Q : \mathbb{R}^n \to \mathbb{R}^n$ is a quadratic form, $B \in \mathrm{Mat}_{n\times n}$. Then

$$\frac{\widetilde{x} - x}{2\epsilon} = Q(x, \widetilde{x}) + \frac{1}{2}B(x + \widetilde{x}),$$

with $Q(x, \widetilde{x}) = \frac{1}{2}\left[Q(x + \widetilde{x}) - Q(x) - Q(\widetilde{x})\right]$ is the symmetric bilinear form corresponding to the quadratic form Q.

✓Equations for $\tilde{x}$ are always linear and the map is always reversible and birational.

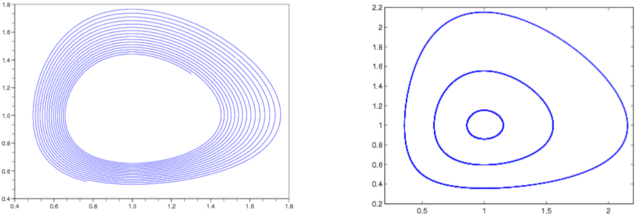
- We use the term Hirota-Kimura (HK) discretization for the Kahan's discretization in the context of integrable systems.

[4]W. Kahan, *Unconventional numerical methods for trajectory calculations*, Unpublished lecture notes, 1993

Two-dimensional integrable systems

Lotka-Volterra system:

$$\begin{cases} \dot{x} = x(1-y), \\ \dot{y} = y(x-1). \end{cases} \qquad \mapsto \qquad \begin{cases} \tilde{x} - x = \epsilon\left(x + \tilde{x} - \tilde{x}y - x\tilde{y}\right), \\ \tilde{y} - y = -\epsilon\left(y + \tilde{y} - \tilde{x}y - x\tilde{y}\right). \end{cases}$$



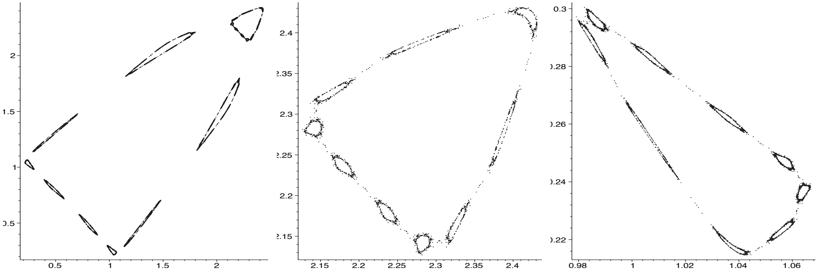
L: one orbit of the explicit Euler method with $\epsilon = 0.01$; R: three orbits of the Kahan's discretization with $\epsilon = 0.1$.

- Non-spiralling behavior.
- Kahan integrator for this system is Poisson [S][5].

[5] J.M. Sanz-Serna, *Appl. Numer. Math.*, 1994, **16**

Two-dimensional integrable systems

1. If $\epsilon = \pm 1$ the map is integrable. If $\epsilon = 1$ it reduces to $\tilde{\tilde{x}}x = \tilde{x}\,(2 - \tilde{x}\,)$ which preserves
$$I = \left[\frac{(x-y)(x+y-2)}{xy} \right]^2 .$$
2. If $\epsilon \neq \pm 1$ an integral of motion (if exists), discretizing $H = x + y - \log(xy)$, is unknown.
3. But numerical experiments indicate non-integrability.



Two-dimensional integrable systems

Weierstrass $\wp$ function ODE:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 6x^2 - g_2/2. \end{cases} \qquad \mapsto \qquad \begin{cases} \tilde{x} - x = \epsilon\,(y + \tilde{y}\,), \\ \tilde{y} - y = \epsilon\,(12x\tilde{x} - g_2)\,. \end{cases}$$

- The first integral of the continuous system reads

$$g_3 = 4x^3 - g_2 x - y^2.$$

- The first integral of the discrete system reads

$$I = \frac{4x^3 - g_2 x - y^2 + 4\epsilon^2 x(g_2 x - y^2) + 4\epsilon^4 g_2^2 x}{1 - 12\epsilon^2 x}.$$

- The discrete map is Poisson:

$$\{x, y\} = 1 - 12\epsilon^2 x.$$

Problem

Let $\dot{x} = f(x)$ be an integrable quadratic vector field. Is its HK discretization integrable?

Conjecture (M. Petrera, A. Pfadler, Yu.B. Suris to appear in *Exp. Math.*, 2008)

For any algebraically completely integrable system with a quadratic vector field, its HK discretization remains algebraically completely integrable.

This is supported by:

- Euler and Lagrange tops (Hirota and Kimura).
- Clebsch system.
- $\mathfrak{so}(3)$ Suslov system (Dragovic and Gajic[6]).
- A class of 3-dimensional bi-Hamiltonian systems.

Preliminary results (with Yu.B. Suris):

- Zhukovsky-Volterra system.
- $\mathfrak{so}(4)$ Euler top.
- Volterra and Toda lattices.
- Classical Gaudin magnet.
- Integrable Henon-Heiles systems.

[6] http://arxiv.org/abs/0807.2966

A flavor of the HK discrete Clebsch system (Yu.B. Suris, A. Pfadler)

The Clebsch system describes the motion of a rigid body in an ideal fluid:

$$\begin{cases} \dot{m}_1 = (\omega_3 - \omega_2) p_2 p_3, \\ \dot{m}_2 = (\omega_1 - \omega_3) p_3 p_1, \\ \dot{m}_3 = (\omega_2 - \omega_1) p_1 p_2, \\ \dot{p}_1 = m_3 p_2 - m_2 p_3, \\ \dot{p}_2 = m_1 p_3 - m_3 p_1, \\ \dot{p}_3 = m_2 p_1 - m_1 p_2. \end{cases}$$

$(\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ parameters, $m = (m_1, m_2, m_3) \in \mathbb{R}^3$, $p = (p_1, p_2, p_3) \in \mathbb{R}^3$.

- It is Hamiltonian w.r.t. Lie-Poisson brackets in $\mathfrak{e}(3)$.
- Four functionally independent integrals of motion:

$$I_i = p_i^2 + \frac{m_j^2}{\omega_k - \omega_i} + \frac{m_k^2}{\omega_j - \omega_i},$$
$$I_4 = m_1 p_1 + m_2 p_2 + m_3 p_3.$$

A flavor of the HK discrete Clebsch system

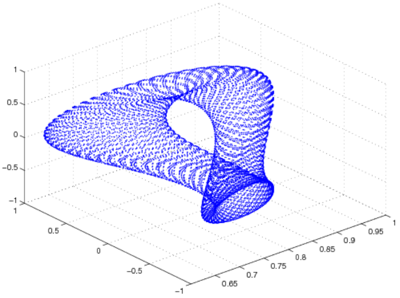
- HK discretization:

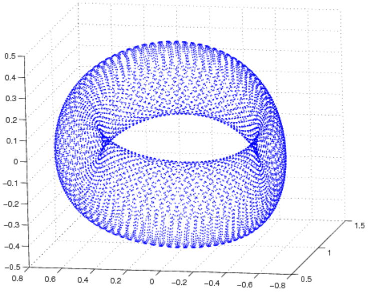
$$\begin{pmatrix} \widetilde{m} \\ \widetilde{p} \end{pmatrix} = f(m,p,\epsilon) = M^{-1}(m,p,\epsilon) \begin{pmatrix} m \\ p \end{pmatrix},$$

$$M(m,p,\epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & \epsilon\omega_{23}p_3 & \epsilon\omega_{23}p_2 \\ 0 & 1 & 0 & \epsilon\omega_{31}p_3 & 0 & \epsilon\omega_{31}p_1 \\ 0 & 0 & 1 & \epsilon\omega_{12}p_2 & \epsilon\omega_{12}p_1 & 0 \\ 0 & \epsilon p_3 & -\epsilon p_2 & 1 & -\epsilon m_3 & \epsilon m_2 \\ -\epsilon p_3 & 0 & \epsilon p_1 & \epsilon m_3 & 1 & -\epsilon m_1 \\ \epsilon p_2 & -\epsilon p_1 & 0 & -\epsilon m_2 & \epsilon m_1 & 1 \end{pmatrix},$$

  with $\omega_{ij} = \omega_i - \omega_j$.
- Numerators and denominators of components of $\tilde{m}$, $\tilde{p}$ are polynomials of degree 6:
  - 31 monomials in numerators of $\tilde{p}_i$
  - 41 monomials in numerators of $\tilde{m}_i$
  - 28 monomials in the common denominator
- The numerator of the $\tilde{\tilde{p}}_1$, as a polynomial of m_k, p_k, ω_k, contains 1.647.595 terms!

A flavor of the HK discrete Clebsch system





Orbits with $\omega_1 = 0.1$, $\omega_2 = 0.2$, $\omega_3 = 0.3$, $\epsilon = 1$; $(m_0, p_0) = (1, 1, 1, 1, 1, 1)$.
L: projections to (m_1, m_2, m_3); R: projections to (p_1, p_2, p_3).

A flavor of the HK discrete Clebsch system

In [PPS][7] we presented an experimental method for a rigorous study of the integrability of HK discretizations.

Goal

- Existence, for every initial point $(m, p) \in \mathbb{R}^6$, of a four-dimensional pencil of quadrics containing the orbit of this point.
- Existence of four functionally independent integrals of motion. There is one simple integral,

$$K = \frac{p_1^2 + p_2^2 + p_3^2}{1 - \epsilon^2(\omega_1 p_1^2 + \omega_2 p_2^2 + \omega_3 p_3^2)}$$

  and three very big integrals.

- This remains true also for an arbitrary flow of the Clebsch system.
- Our proofs are computer assisted. A general structure behind the integrability of the HK mechanism remains unknown.
- Nothing is known about the Lax representation and the existence of an invariant Poisson structure.

[7] M. Petrera, A. Pfadler, Yu.B. Suris to appear in *Exp. Math.*, 2008

3-dimensional HK-type discrete systems (A.N.W. Hone)

- The Euler top is the most famous three-dimensional bi-Hamiltonian system. Its HK discretization provides a completely integrable discrete-time system.
- In [GN][8] the authors construct a list of all (12) non-trivial bi-Hamiltonian flows associated with pairs of real 3-dim. Lie algebras:

$$PdH = 0, \qquad \dot{\mathbf{x}} = (\dot{x}, \dot{y}, \dot{z}) = -PdK = -\frac{1}{c}QdH, \qquad QdK = 0,$$

$c = c(x, y, z)$ conformal factor. Each flow preserves the measure form

$$\omega = c\, dx \wedge dy \wedge dz.$$

Aim

To construct the HK discretization of GN bi-Hamiltonian quadratic vector fields associated with real 3-dim. Lie algebras

- To begin with, we shall be concerned with only 6 out of the 12 flows, the ones which have non-trascendental integrals of motions.

[8] H. Gümral, Y. Nutku, *Journ. Math. Phys.*, 1993, **34**

GN bi-Hamiltonian flows with non-transcendental invariants:

$$
\begin{array}{lllll}
\mathcal{E}_1: & \dot{x} = -x^2, & \dot{y} = -xy, & \dot{z} = 2y^2 + xz, & (P^{(1)}, c_1^{-1} Q^{(1)}), \\
\mathcal{E}_2: & \dot{x} = -x^2, & \dot{y} = xy, & \dot{z} = -2y^2 + xz, & (P^{(2)}, c_2^{-1} Q^{(2)}), \\
\mathcal{E}_3: & \dot{x} = -xz, & \dot{y} = -yz, & \dot{z} = x^2 + y^2, & (P^{(3)}, c_3^{-1} Q^{(3)}), \\
\mathcal{E}_4: & \dot{x} = -xz, & \dot{y} = yz, & \dot{z} = x^2 - y^2, & (P^{(4)}, c_4^{-1} Q^{(4)}), \\
\mathcal{E}_5: & \dot{x} = xy, & \dot{y} = -x^2, & \dot{z} = y(2x - z), & (P^{(5)}, c_5^{-1} Q^{(5)}), \\
\mathcal{E}_6: & \dot{x} = y(2z - x), & \dot{y} = x^2 - z^2, & \dot{z} = y(z - 2x). & (P^{(6)}, c_6^{-1} Q^{(6)}).
\end{array}
$$

- For instance, $\mathcal{E}_1$ is bi-Hamiltonian w.r.t. $(P^{(1)}, c_1^{-1} Q^{(1)})$,

$$
\begin{array}{lll}
P^{(1)}_{12} = \{x, y\} = 0, & P^{(1)}_{23} = \{y, z\} = y, & P^{(1)}_{31} = \{z, x\} = -x, \\
Q^{(1)}_{12} = \{x, y\} = x, & Q^{(1)}_{23} = \{y, z\} = z, & Q^{(1)}_{31} = \{z, x\} = -2y,
\end{array}
$$

$$
P^{(1)} dH_1 = 0, \qquad P^{(1)} dK_1 = \frac{1}{c_1} Q^{(1)} dH_1 \qquad Q^{(1)} dK_1 = 0, \qquad c_1 = \frac{1}{x^2},
$$

with $H_1 = y/x$ and $K_1 = zx + y^2$.

- $\mathcal{E}_6$ corresponds to a particular case of of the $\mathfrak{so}(3)$ Euler top.

i	$P_{12}^{(i)}$	$P_{23}^{(i)}$	$P_{31}^{(i)}$	$Q_{12}^{(i)}$	$Q_{23}^{(i)}$	$Q_{31}^{(i)}$	H_i	K_i	c_i
1	0	y	$-x$	x	z	$2y$	$\frac{y}{x}$	$zx+y^2$	$\frac{1}{x^2}$
2	0	$-y$	$-x$	x	z	$2y$	xy	$zx+y^2$	1
3	0	y	$-x$	z	x	y	$\frac{y}{x}$	$\frac{1}{2}(x^2+y^2+z^2)$	$\frac{1}{x^2}$
4	0	$-y$	$-x$	z	x	y	xy	$\frac{1}{2}(x^2+y^2+z^2)$	1
5	0	x	y	x	z	$2y$	$\frac{1}{2}(x^2+y^2)$	$zx+y^2$	-1
6	x	z	$2y$	z	x	y	$zx+y^2$	$\frac{1}{2}(x^2+y^2+z^2)$	-1

There are just 5 independent Lie-Poisson structures,

$$P^{(1)} = P^{(3)},\ P^{(2)} = P^{(4)},\ P^{(5)},\ P^{(6)} = Q^{(1)} = Q^{(2)} = Q^{(5)},\ Q^{(3)} = Q^{(4)} = Q^{(6)},$$

corresponding to $A_{3,3}, \mathfrak{e}(1,1), \mathfrak{e}(2), \mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(3)$ and with Casimir functions

$$H_1 = H_3,\ H_2 = H_4,\ H_5,\ H_6 = K_1 = K_2 = K_5,\ K_3 = K_4 = K_6.$$

HK discretization

Let $\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$. Then:

$$\dot{x}_i \mapsto \frac{\tilde{x}_i - x_i}{2\epsilon}, \qquad x_j x_k \mapsto \frac{\tilde{x}_j x_k + x_j \tilde{x}_k}{2}.$$

Theorem. The HK discretizations of the quadratic vector fields $\mathcal{E}_i$, $1 \leq i \leq 6$, read

$$d\mathcal{E}_i : \qquad \tilde{\mathbf{x}} = A_i^{-1}(\mathbf{x}; \epsilon)\mathbf{x} = A_i(\tilde{\mathbf{x}}; -\epsilon)\mathbf{x}.$$

The maps $d\mathcal{E}_i$ are completely integrable. In particular:

1. They have two integrals of motion, $H_i(\epsilon), K_i(\epsilon)$.
2. They preserve the volume form

$$\Omega_i = \frac{c_i}{H_i K_i} dx \wedge dy \wedge dz, \qquad 1 \leq i \leq 6.$$

3. They admit the compatible invariant Poisson structures $(P^{(i)}(\epsilon), c_i^{-1} Q^{(i)}(\epsilon))$.
4. They admit explicit solutions.

i	$A_i(\mathbf{x};\epsilon)$	$H_i(\epsilon)$	$K_i(\epsilon)$
1	$\begin{pmatrix} 1+2\epsilon x & 0 & 0 \\ \epsilon y & 1+\epsilon x & 0 \\ -\epsilon z & -4\epsilon y & 1-\epsilon x \end{pmatrix}$	H_1	$\dfrac{K_1}{1-\epsilon^2 x^2}$
2	$\begin{pmatrix} 1+2\epsilon x & 0 & 0 \\ -\epsilon y & 1-\epsilon x & 0 \\ -\epsilon z & 4\epsilon y & 1-\epsilon x \end{pmatrix}$	$\dfrac{H_2}{1-\epsilon^2 x^2}$	$\dfrac{K_2}{1-\epsilon^2 x^2}$
3	$\begin{pmatrix} 1+\epsilon z & 0 & \epsilon x \\ 0 & 1+\epsilon z & \epsilon y \\ -2\epsilon x & -2\epsilon y & 1 \end{pmatrix}$	H_3	$\dfrac{K_3}{1+\epsilon^2(x^2+y^2)}$
4	$\begin{pmatrix} 1+\epsilon z & 0 & \epsilon x \\ 0 & 1-\epsilon z & -\epsilon y \\ -2\epsilon x & 2\epsilon y & 1 \end{pmatrix}$	$\dfrac{H_4}{1+\epsilon^2(x^2+y^2)}$	$\dfrac{K_4}{1+\epsilon^2(x^2+y^2)}$
5	$\begin{pmatrix} 1-\epsilon y & -\epsilon x & 0 \\ 2\epsilon x & 1 & 0 \\ -2\epsilon y & -\epsilon(2x-z) & 1+\epsilon y \end{pmatrix}$	$\dfrac{H_5}{1+\epsilon^2 x^2}$	$\dfrac{K_5}{1+\epsilon^2 x^2}$
6	$\begin{pmatrix} 1+\epsilon y & \epsilon(x-2z) & -2\epsilon y \\ -2\epsilon x & 1 & 2\epsilon z \\ 2\epsilon y & \epsilon(2x-z) & 1-\epsilon y \end{pmatrix}$	$\dfrac{H_6}{1-3\epsilon^2 xz}$	$\dfrac{K_6}{1-3\epsilon^2 xz}$

First deformed Poisson structure:

i	$P_{12}^{(i)}(\epsilon)$	$P_{23}^{(i)}(\epsilon)$	$P_{31}^{(i)}(\epsilon)$
1	0	$y\left(1-\epsilon^2x^2\right)$	$-x\left(1-\epsilon^2x^2\right)$
2	0	$-y\left(1+\epsilon^2x^2\right)$	$-x\left(1-\epsilon^2x^2\right)$
3	0	$y\left[1+\epsilon^2(x^2+y^2)\right]$	$-x\left[1+\epsilon^2(x^2+y^2)\right]$
4	0	$-y\left[1-\epsilon^2(x^2-y^2)\right]$	$-x\left[1+\epsilon^2(x^2-y^2)\right]$
5	0	$x\left(1-\epsilon^2y^2\right)$	$y\left(1+\epsilon^2x^2\right)$
6	$x\left(1+3\epsilon^2y^2\right)$	$z\left(1+3\epsilon^2y^2\right)$	$2y\left(1-3\epsilon^2xz\right)$

Second deformed Poisson structure:

i	$Q_{12}^{(i)}(\epsilon)$	$Q_{23}^{(i)}(\epsilon)$	$Q_{31}^{(i)}(\epsilon)$
1	x	$\dfrac{z+\epsilon^2 x(zx+2y^2)}{1-\epsilon^2 x^2}$	$2y$
2	$x\left(1-\epsilon^2x^2\right)$	$z+\epsilon^2 x(zx+2y^2)$	$2y\left(1-\epsilon^2x^2\right)$
3	z	$\dfrac{x\left(1-\epsilon^2 z\right)}{1+\epsilon^2(x^2+y^2)}$	$\dfrac{y\left(1-\epsilon^2 z\right)}{1+\epsilon^2(x^2+y^2)}$
4	$z\left[1+\epsilon^2(x^2+y^2)\right]$	$x\left(1-\epsilon^2z^2\right)$	$y\left(1-\epsilon^2z^2\right)$
5	$x\left(1+\epsilon^2x^2\right)$	$z-\epsilon^2x(xz+2y^2)$	$2y\left(1+\epsilon^2x^2\right)$
6	$z+\frac{3}{2}\epsilon^2x(x^2+y^2-z^2)$	$x+\frac{3}{2}\epsilon^2z(z^2+y^2-x^2)$	$y\left(1-3\epsilon^2xz\right)$

For instance, $i = 4$:

$$\begin{cases} \dot{x} = -zx, \\ \dot{y} = xy, \\ \dot{z} = x^2 - y^2, \end{cases} \qquad \mapsto \qquad \begin{cases} \tilde{x} - x = -\epsilon(\tilde{z}x + z\tilde{x}), \\ \tilde{y} - y = \epsilon(\tilde{y}x + y\tilde{x}), \\ \tilde{z} - z = 2\epsilon(\tilde{x}x - \tilde{y}y), \end{cases}$$

$$H_4(\epsilon) = \frac{xy}{1 + \epsilon^2(x^2 + y^2)}, \qquad K_4(\epsilon) = \frac{1}{2}\,\frac{x^2 + y^2 + z^2}{1 + \epsilon^2(x^2 + y^2)},$$

$$\{x, y\} = 0, \quad \{y, z\} = -y\left[1 - \epsilon^2(x^2 - y^2)\right], \quad \{z, x\} = -x\left[1 + \epsilon^2(x^2 - y^2)\right],$$

$$\{x, y\} = z\left[1 + \epsilon^2(x^2 + y^2)\right], \quad \{y, z\} = x\left(1 - \epsilon^2 z^2\right), \quad \{z, x\} = y\left(1 - \epsilon^2 z^2\right).$$

$$\begin{cases} x_n = sn\,(v/2)\left[\dfrac{dn(u + nv)}{cn(v/2)} + \dfrac{k\,cn(u + nv)}{dn(v/2)}\right], \\ y_n = sn\,(v/2)\left[\dfrac{dn(u + nv)}{cn(v/2)} - \dfrac{k\,cn(u + nv)}{dn(v/2)}\right], \\ z_n = 2\,k\,sn\,(v/2)\,sn(u + nv), \end{cases}$$

u and v expressed in terms of the first integrals.

Systems with one transcendental invariant

We considered 2 GN systems with one transcendental invariants:

$$\mathcal{E}_7: \quad \dot{x} = -x^2, \quad \dot{y} = -\xi xy, \quad \dot{z} = 2\xi y^2 + xz,$$
$$\mathcal{E}_8: \quad \dot{x} = -x^2, \quad \dot{y} = -x(x+y), \quad \dot{z} = 2y(x+y) + xz,$$

with $|\xi| \in (0,1)$. Note that $\mathcal{E}_7\big|_{\xi=1} = \mathcal{E}_1$ and $\mathcal{E}_7\big|_{\xi=-1} = \mathcal{E}_2$.

$$P^{(7)} dH_7 = 0, \qquad P^{(7)} dK_7 = \frac{1}{c_7} Q^{(7)} dH_7, \qquad Q^{(7)} dK_7 = 0,$$

$$Q^{(7)} = P^{(6)}, \qquad P^{(7)}_{12} = 0, \qquad P^{(7)}_{23} = \xi y, \qquad P^{(7)}_{31} = -x,$$

$$H_7 = y x^{-\xi}, \qquad K_7 = H_6 = zx + y^2, \qquad c_7 = x^{-(\xi+1)},$$

$$x(t) = \frac{1}{t+\alpha}, \qquad y(t) = \beta (t+\alpha)^{-\xi}, \qquad z(t) = (t+\alpha)\left[\gamma - \beta^2 (t+\alpha)^{-2\xi}\right],$$

with $H_7 = \beta$ and $K_7 = \gamma$.

Explicit solutions of $d\mathcal{E}_7$ ($\epsilon = 1/2$):

$$x_n = \frac{1}{n+\alpha}, \tag{1}$$

$$y_n = \frac{\alpha\beta}{n+\alpha} \frac{\Gamma\left(n+1+\alpha-\xi/2\right)\Gamma\left(\alpha+\xi/2\right)}{\Gamma\left(n+\alpha+\xi/2\right)\Gamma\left(\alpha+1-\xi/2\right)}, \tag{2}$$

$$z_n = \frac{\alpha\gamma[4(n+\alpha)^2-1]}{(4\alpha^2-1)(n+\alpha)} + \frac{4\alpha^2\beta^2\xi[4(n+\alpha)^2-1]\Gamma^2\left(\alpha+\xi/2\right)}{\Gamma^2\left(\alpha+1-\xi/2\right)(n+\alpha)} W_n, \tag{3}$$

with

$$W_n = \sum_{j=0}^{n-1} \frac{[2(j+1+\alpha)-\xi]\Gamma^2\left(j+1+\alpha-\xi/2\right)}{[2(j+\alpha)+3][2(j+\alpha)+\xi][4(j+\alpha)^2-1]\Gamma^2\left(j+\alpha+\xi/2\right)}.$$

From (1) and (2) one gets:

$$\widehat{H}_7 = \frac{y_n}{x_n} \frac{\Gamma\left(x_n^{-1}+\xi/2\right)}{\Gamma\left(x_n^{-1}+1-\xi/2\right)}.$$

The second invariant is given implicitly by (3).

Diophantine integrability

- Diophantine integrability test [H][9]: simple criterion to detect the integrability of rational maps over $\mathbb{Q}$.
- For a map whose n-th iterate has components $x_n = p_n/q_n \in \mathbb{Q}$, the height of x_n is $H(x_n) = max(|p_n|, |q_n|)$ and the logarithmic height is $h(x_n) = \log H(x_n)$.
- A map is Diophantine integrable if h_n of the iterates of all orbits has at most polynomial growth in n:

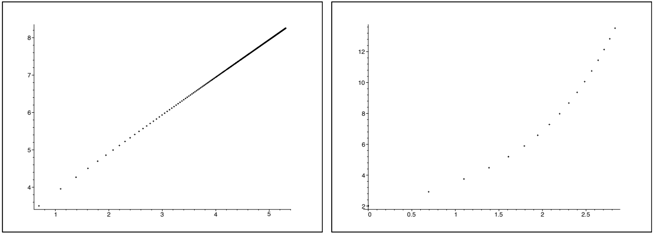
$$E(\mathcal{O}) = 0, \quad \forall \mathcal{O}, \qquad E(\mathcal{O}) = \lim_{n\to\infty} \frac{1}{n} \log[h(x_n)].$$

- It is similar to algebraic entropy: measure of the height growth of rational functions generated by rational maps. The height is the maximum of the degrees of the polynomials in the numerator and denominator as functions of initial data.

[9] R. Halburd, *Journ. Phys. A*, 2005, **38**

Diophantine integrability

- Disadvantage of algebraic entropy: to guess a recursive relation to generate the degrees of polynomials
- Advantage of Diophantine entropy: quick numerical implementation. If a map is Diophantine integrable then an plot of $\log h(x_n)$ vs. $\log n$ looks asymptotically like a straight line, otherwise it will have an exponential growth.



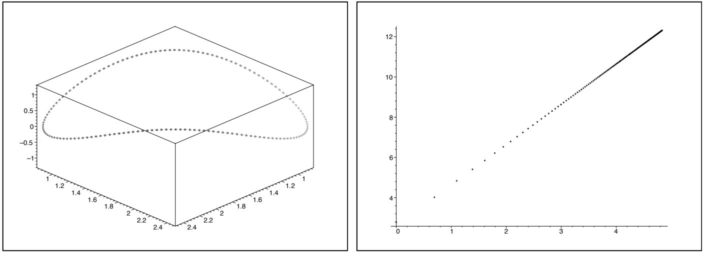
L: 200 iterates of $d\mathcal{E}_3$; R: 17 iterates of dLV.

- But it is no clear how the Diophantine integrability is related with other definitions of integrability.
- For 2-dim. and 3-dim. maps Diophantine integrability is a necessary condition for algebraic integrability.

Diophantine integrability

Theorem. The discrete systems $d\mathcal{E}_i$, $1 \leq i \leq 8$, are all Diophantine integrable.

- The above claim may be easily supported by numerical observations.



L: 1000 iterates of $d\mathcal{E}_4$; R: $\log h(x_n)$ vs. $\log n$ for 125 iterates of $d\mathcal{E}_4$.

- $d\mathcal{E}_7$ and $d\mathcal{E}_8$ provide the first examples of discrete integrable systems with transcendental invariants that are also Diophantine integrable.

Concluding remarks

Goals

- Conjecture. For any algebraically completely integrable system with a quadratic vector field, its HK discretization remains algebraically completely integrable.
- We used the HK procedure to construct several integrable discrete systems.
- Huge advantage: the procedure is systematic and algorithmic.

Open problems (work in progress)

- Preliminary results:
  - Zhukovsky-Volterra system.
  - $\mathfrak{so}(4)$ Euler top.
  - Volterra and Toda lattices.
  - Classical Gaudin magnet.
  - Integrable Henon-Heiles systems.
- There should be some deep mathematical structure behind the HK discretization. If our conjecture is true it should be related to addition theorems for multi-dimensional theta functions.