On q-deformed Whittaker functions

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2. A. Gerasimov, D. Lebedev, S. Oblezin *Baxter Q-operators and their Arithmetic implications*, To appear in M. Duflo volume, Lett. Math. Phys. (2008)
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- **Wittaker functions.** Let G be a (quasi)-split real connected reductive group. The Iwasawa decomposition:

$$G = U \cdot A \cdot K \qquad\qquad A = \left\{ e^{x_1}, \ldots, e^{x_N} \right\}$$

In the case $G = GL(N, \mathbb{R})$ we have $U = \|n_{ij}\|_{i<j}$.

Definition 1 *The Whittaker function $\Psi_{\underline{\gamma}}(z)$ is a smooth function on symmetric space X (either $X = G/K$, or $X = G/B$ for $B = UA$) analytic in $\underline{\gamma}$, such that for any $n \in U$*

$$\Psi_{\underline{\gamma}}(nz) \;=\; \exp\left\{ 2\pi \imath \sum_{i=1}^{N-1} n_{i,i+1} \right\} \Psi_{\underline{\gamma}}(z)$$

and

$$\mathcal{H}_r(\underline{x}) \cdot \Psi_{\underline{\gamma}}(\underline{x}) \;=\; \left(\frac{1}{r!} \sum_{i=1}^{N} \gamma_i^r \right) \Psi_{\underline{\gamma}}(\underline{x})$$

1. The function $\Psi_{\underline{\gamma}}(z)$ is the Fourier coefficient of Maaß form;
2. $\Psi_{\underline{\gamma}}(z)$ is the matrix element of (infinite-dimensional) principal series representation $\mathrm{Ind}_B^G \chi_{\underline{\gamma}}$;
3. For p-adic Whittaker function the Shintani-Casselman-Shalika formula holds:

$$\Psi_{\underline{\gamma}}^{\mathfrak{g}(\mathbb{Q}_p)}(\underline{x}) \;=\; \mathrm{Tr}_V \prod_{i=1}^{N} z_i^{E_{ii}}$$

4. (B. Kostant) $\Psi_{\underline{\gamma}}(z)$ is the Toda chain wave function
5. (A. Givental) The q-deformed Whittaker function equals an equivariant Euler characteristic of the loop space $L_+X = \mathrm{Map}^+(S^1,\, X)$.

• In 1996 Givental proposed an integral formula for the common eigenfunction of $GL(N,\mathbb{R})$-Toda Hamiltonians:

$$\mathcal{H} \;=\; -\frac{\hbar^2}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} \;+\; \sum_{i=1}^{N-1} e^{x_i - x_{i+1}}\,.$$

$$\boxed{\Psi_{\gamma_1,\ldots,\gamma_N}(x_{N,1},\ldots,x_{N,N}) \;=\; \int_{\mathcal{C}} \quad e^{\frac{1}{\hbar}\,\mathcal{F}_N(x)} \quad \prod_{k=1}^{N-1}\prod_{i=1}^{k} dx_{k,i},}$$

where $T_{N,i} := T_i, \quad \mathcal{C} = \mathbb{R}^{N(N-1)/2} \subset \{\exp(T_{k,i})\}$

$$\mathcal{F}_N(T) = \imath\sum_{k=1}^{N}\gamma_k\Big(\sum_{i=1}^{k}x_{k,i} - \sum_{i=1}^{k-1}x_{k-1,i}\Big) - \sum_{k=1}^{N-1}\sum_{i=1}^{k}\left(e^{x_{k+1,i}-x_{k,i}} + e^{x_{k,i}-x_{k+1,i+1}}\right)$$

$$\begin{array}{ccccccc}
x_{N,1} & & & & & & \\
\uparrow & & & & & & \\
x_{N-1,1} & \longleftarrow & x_{N,2} & & & & \\
\uparrow & & \uparrow & & & & \\
\vdots & & \vdots & & & & \\
x_{2,1} & \longleftarrow & x_{3,2} & \longleftarrow & \ldots\; x_{N,N-1} & & \\
\uparrow & & \uparrow & & \uparrow & & \\
x_{1,1} & \longleftarrow & x_{2,2} & \longleftarrow & \ldots x_{N-1,N-1} & \longleftarrow & x_{NN}
\end{array}$$

To an arrow $x \longrightarrow x'$ the function $e^{x'-x}$ is assigned.

• In 2000 Kharchev and Lebedev introduced another integral representation (of Mellin-Barnes type) of $GL(N,\mathbb{R})$-Whittaker function.

$$\Psi^{\mathfrak{gl}_N}_{\underline{\gamma}}(\underline{x}) = \int_{\mathcal{S}} \prod_{n=1}^{N-1} \frac{\prod\limits_{m=1}^{n+1}\prod\limits_{k=1}^{n} \Gamma\left(\frac{\imath\gamma_{n+1,m}-\imath\gamma_{n,k}}{\hbar}\right)}{\prod\limits_{s\neq p} \Gamma\left(\frac{\imath\gamma_{n,s}-\imath\gamma_{n,p}}{\hbar}\right)} \cdot$$

$$\cdot \exp\left\{\imath x_n\left(\sum_{j=1}^{N}\gamma_{n,j} - \sum_{j=1}^{N}\gamma_{n-1,j}\right)\right\} \prod_{\substack{n=1\\ j\leq n}}^{N-1} d\gamma_{nj}$$

where

$$\underline{\gamma} = (\gamma_1,\ldots,\gamma_N) = (\gamma_{N,1},\ldots,\gamma_{NN})$$
$$\underline{x} = (x_1,\ldots,x_N) = (x_{N,1},\ldots,x_{NN})$$

Domain of the integration $\mathcal{S}$ is defined by

$$\min_j\{\operatorname{Im}\gamma_{kj}\} > \max_m\{\operatorname{Im}\gamma_{k+1,m}\} \qquad k = 1,\ldots,N-1$$

The **Gelfand-Zetlin pattern** is

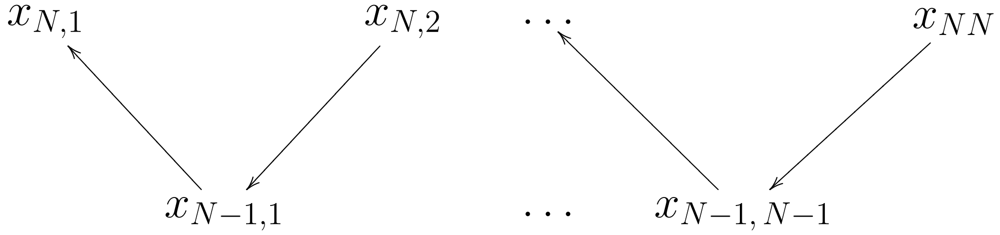
$$\begin{array}{ccccccccc}
\gamma_{N,1} & & \gamma_{N,2} & & \cdots & & \gamma_{N,N-1} & & \gamma_{NN} \\
& \gamma_{N-1,1} & & \gamma_{N-1,2} & & \cdots & & \gamma_{N-1,N-1} & \\
& & \ddots & & \cdots & & \cdots & & \\
& & & \gamma_{21} & & \gamma_{22} & & & \\
& & & & \gamma_{11} & & & &
\end{array}$$

- Introduce a pair of (dual) recursive integral operators:

$$Q^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}}(\underline{x}_N\,;\,\underline{x}_{N-1}|\,\gamma_N)\;=\;\exp\Big\{\,\imath\gamma_N\Big(\sum_{i=1}^{N}x_{N,i}-\sum_{i=1}^{N-1}x_{N-1,i}\Big)-$$

$$-\sum_{i=1}^{N-1}\Big(e^{x_{N,i}-x_{N-1,i}}+e^{x_{N-1,i}-x_{N,i+1}}\Big)\Big\}$$

corresponding to Givental diagram



$$^{\vee}Q^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}}(\underline{\gamma}_N,\underline{\gamma}_{N-1}|\,x_N)\;=\;\prod_{j=1}^{N}\prod_{k=1}^{N-1}\Gamma\Big(\frac{\imath\gamma_{N,j}-\imath\gamma_{N-1,k}}{\hbar}\Big)\cdot$$

$$\cdot\exp\Big\{\imath x_N\Big(\sum_{j=1}^{N}\gamma_{N,j}\;-\;\sum_{k=1}^{N-1}\gamma_{N-1,k}\Big)\Big\}$$

corresponding to the Gelfand-Tsetlin pattern

$\gamma_{N,1}$ $\quad$ $\gamma_{N,2}$ $\quad$ $\cdots$ $\quad$ $\gamma_{N,N-1}$ $\quad$ γ_{NN}

$\gamma_{N-1,1}$ $\quad$ $\gamma_{N-1,2}$ $\quad$ $\cdots$ $\quad$ $\gamma_{N-1,N-1}$

• **Macdonald polynomials.** Let $\underline{\lambda} = (\lambda_1 \geq \ldots \geq \lambda_N)$ be a partition, then define

$$\mu_{\underline{\lambda}} = \sum_{\sigma \in \mathfrak{S}_N} x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdot \ldots \cdot x_{\sigma(N)}^{\lambda_N}$$

$$\pi_{\underline{\lambda}} = \prod_{i=1}^{N} \pi_{\lambda_i} \qquad \pi_n = \sum_{i=1}^{N} x_i^n$$

On symmetric functions over $\mathbb{Q}(q,t)$ define a pairing by

$$\left\langle \pi_{\underline{\lambda}},\, \pi_{\underline{\lambda}'} \right\rangle_{q,t} = \delta_{\underline{\lambda},\,\underline{\lambda}'} \prod_{n \geq 1} n^{m_n} m_n! \prod_{\lambda_k \neq 0} \frac{1 - q^{\lambda_k}}{1 - t^{\lambda_k}}$$

where $m_n = \big|\{k \,|\, \lambda_k = n\}\big|$.

Definition 2 *Macdonald polynomials* $P_{\underline{\lambda}}(\underline{x}|q,t)$ *are the symmetric functions over* $\mathbb{Q}(q,t)$ *such that*

$$P_{\underline{\lambda}}(\underline{x}|q,t) = \mu_\lambda + \sum_{\underline{\lambda}' \preceq \underline{\lambda}} u_{\underline{\lambda}',\,\underline{\lambda}}\, \mu_{\underline{\lambda}'} \qquad u_{\underline{\lambda}',\,\underline{\lambda}} \in \mathbb{Q}(q,t)$$

$$\left\langle P_{\underline{\lambda}}(\underline{x}),\, P_{\underline{\lambda}'}(\underline{x}) \right\rangle_{q,t} = 0 \qquad \underline{\lambda}' \neq \underline{\lambda}$$

Macdonald polynomials satisfy the following basic properties:

$$\prod_{i=1}^{N} \prod_{j=1}^{N} \prod_{n=0}^{\infty} \frac{1 - t x_i y_j q^n}{1 - x_i y_j q^n} = \sum_{\underline{\lambda}} P_{\underline{\lambda}}(\underline{x})\, P_{\underline{\lambda}}^*(\underline{y}) \tag{1}$$

and

$$H_r\, P_{\underline{\lambda}}(\underline{x}) = c_r(q^{\underline{\lambda}})\, P_{\underline{\lambda}}(\underline{x}) \qquad c_r(q^{\underline{\lambda}}) = \sum_{I_r} \prod_{i \in I_r} q^{\lambda_i}\, t^{\varrho_i}$$

$$H_r = \sum_{I_r} t^{r(r-1)/2} \prod_{i \in I_r,\, j \notin I_r} \frac{t x_i - x_j}{x_i - x_j} \prod_{i \in I_r} T_{x_i}, \tag{2}$$

where $I_r = (i_1 < i_2 < \ldots < i_r)$ for $r = 1, \ldots, N$.

• **q-Whittaker function: The first formula.** Let $q < 1$, consider the limit $t = q^{-k} \to \infty$, $k \to \infty$ of $P_{\underline{\lambda}}(\underline{x}|q,t)$

$$\mathcal{P}^{(N)} := \left\{ p_{k,i} \,\middle|\, p_{k+1,i} \geq p_{k,i} \geq p_{k+1,i+1} \right\} \subset \mathbb{Z}^{N(N-1)/2}$$

$$\mathcal{P}_{N,N-1} := \left\{ \underline{p}_{N-1} \,\middle|\, p_{N,i} \geq p_{N-1,i} \geq p_{N,i+1} \right\} \subset \mathcal{P}^{(N)}$$

Theorem 1 *The q-Whittaker function ${}^q\Psi_{\underline{z}}(\underline{p}_N)$ reads*
(I) For $\underline{p}_N$ being in the dominant domain $p_{N,1} \geq \ldots \geq p_{NN}$

$$ {}^q\Psi_{\underline{z}}(\underline{p}_N) = \sum_{\underline{p}_k \in \mathcal{P}^{(N)}} \prod_{k=1}^{N} z_k^{|\underline{p}_k| - |\underline{p}_{k-1}|}$$

$$\cdot \frac{\prod\limits_{k=2}^{N-1} \prod\limits_{i=1}^{k-1} (p_{k,i} - p_{k,i+1})_q!}{\prod\limits_{1 \leq i \leq k \leq N-1} (p_{k+1,i} - p_{k,i})_q! \, (p_{k,i} - p_{k+1,i+1})_q!},$$

(II) When $\underline{p}_N$ is outside the dominant domain ${}^q\Psi_{\underline{z}}(\underline{p}) = 0$.

This formula admits a recursion with respect to the rank N:

$$ {}^q\Psi_{\underline{z}}(\underline{p}_N) = \sum_{\underline{p}_{N-1} \in \mathcal{P}_{N,\,N-1}} \Delta(\underline{p}_{N-1}) \; z_N^{|\underline{p}_N| - |\underline{p}_{N-1}|}$$

$$\cdot \; Q^{\mathfrak{gl}_N}_{\mathfrak{gl}(N-1)}(\underline{p}_N, \underline{p}_{N-1} | \, q) \;\; {}^q\Psi_{\underline{z}'}(\underline{p}_{N-1}),$$

where $\underline{z}' = (z_1, \ldots, z_{N-1})$, and

$$Q^{\mathfrak{gl}_N}_{\mathfrak{gl}(N-1)}(\underline{p}_N, \underline{p}_{N-1} | \, q) = \frac{1}{\prod\limits_{i=1}^{N-1} (p_{N,i} - p_{N-1,i})_q! \, (p_{N-1,i} - p_{N,i+1})_q!}$$

$$\Delta(\underline{p}_{N-1}) = \prod_{i=1}^{N-2} (p_{N-1,i} - p_{N-1,i+1})_q!$$

• In the limit $q \longrightarrow 0$ when $p_{N,1} \geq \ldots \geq p_{NN}$

$$\lim_{q\to 0} {}^q\Psi_{\underline{z}}(\underline{p}) \;=\; \sum_{\underline{p}_k \in \mathcal{P}^{(N)}} \prod_{k=1}^{N} z_k^{|\underline{p}_k| - |\underline{p}_{k-1}|} \;=\; \chi_{\underline{p}}(\underline{z})$$

• In the limit $q \longrightarrow 1$ consider ${}^q\widetilde{\Psi}_{\underline{z}}(\underline{p}) := \Delta(\underline{p}_N)\, {}^q\Psi_{\underline{z}}(\underline{p})$

$$\lim_{q\to 1} {}^q\widetilde{\Psi}_{\underline{z}}(\underline{p}) \;=\; \chi_N(\underline{z})^{p_{NN}} \prod_{k=1}^{N-1} \chi_k(\underline{z})^{p_{N,k} - p_{N,k+1}} \;=\; \mathrm{Tr}_{\,V_{\mathrm{f}}} \prod_{i=1}^{N} z_i^{E_{ii}}$$

$$V_{\mathrm{f}} \;=\; V_{\varpi_N}^{\otimes p_{NN}} \bigotimes_{k=1}^{N-1} V_{\varpi_k}^{\otimes(p_{N,k} - p_{N,k+1})}$$

• For $w \in \widehat{W}$ the subspace $V^{[w\cdot\varpi]}(\varpi) \subset V(\varpi)$ is 1-dim. Demazure module $V_w(\varpi) := U(\mathfrak{b}) \cdot V^{[w\cdot\varpi]}(\varpi) \subset V(\varpi)$

$$\mathrm{ch}_{\;V_w(\varpi)} \;=\; \sum_{\mu \in P} \dim V_w^{[\mu]}(\varpi)\, e^{\mu} \;=\; \mathcal{D}_{s_{i_1}} \cdot \ldots \cdot \mathcal{D}_{s_{i_m}} \cdot e^{\varpi}$$

where $\mathcal{D}_{s_i} \;=\; \frac{1 - e^{-\alpha_i}\, s_i}{1 - e^{-\alpha_i}}$, and $w \;=\; s_{i_1} \cdots s_{i_m}$. Identify a set of $\widehat{W}$-orbits in $\dot{P}$ with $\mathbb{Z} \times (\mathbb{Z}/\mathbb{Z}_N)$, and for $0 < i < N$ let

$$\varpi_{k,i} \;:=\; k\varpi_N \;+\; \varpi_i \;=\; (\underbrace{k+1, \ldots, k+1}_{i}, k, \ldots, k)$$

be the representatives of the $\widehat{W}$-orbits. Besides, let us introduce the homomorphism $\pi :\; \mathbb{Z}[A] \longrightarrow \mathbb{Z}[q; z_1, \ldots, z_N]$ defined by $\pi(e^{\varpi_0}) = 1$, $\pi(e^{\varpi_k}) = z_1 \cdots z_k$, $0 < k \leq N$, and $\pi(e^{\delta}) = q$.

Theorem 2 *Let $\widehat{\varpi}_{k,i} := \varpi_0 + \varpi_{k,i}$, and let $w_{k,i} \in \widehat{W}$ be such that the projection of $w_{k,i} \cdot \widehat{\varpi}_{k,i}$ onto P is anti-dominant. Let $\Lambda_{k,i} := w_0 \cdot \varpi_{k,i}$; then the following holds.*

$${}^q\widetilde{\Psi}_{\underline{z}}(\underline{p}_N) \;=\; q^{\frac{1}{2}(\varpi_{k,i},\, \varpi_{k,i}) - \frac{1}{2}(\Lambda_{k,i},\, \Lambda_{k,i})}\, \pi\big(\, \mathrm{ch}_{\;V_{w_{k,i}}(\Lambda_{k,i})}\big)$$

where $\underline{p}_N = \Lambda_{k,i}$.

• **q-Whittaker function: The second formula.** Consider the limit $t \to 0$. Let $\{z_{k,i};\, 1 \leq i \leq k \leq N\}$ with $z_{N,i} := z_i$, $1 \leq i \leq N$.

Theorem 3 *In the dominant domain $p_{N,1} \geq \ldots \geq p_{NN}$*

$$
{}^{q}\Psi_{\underline{z}_N}(\underline{p}_N) \,=\, \Gamma_q(q)^{\frac{(N-1)(N-2)}{2}} \prod_{1\leq j\leq n\leq N-1} \oint \frac{dz_{n,j}}{2\pi\imath z_{n,j}}
$$

$$
\cdot \prod_{1\leq i\leq k\leq N} \Big(\frac{z_{k,i}}{z_{k-1,i}}\Big)^{p_{N,k}} \prod_{n=1}^{N-1} \frac{\prod\limits_{i=1}^{n+1}\prod\limits_{j=1}^{n} \Gamma_q\big(z_{n,j}^{-1} z_{n+1,i}\big)}{n!\, \prod\limits_{j\neq m} \Gamma_q\big(z_{n,m}^{-1} z_{n,j}\big)}
$$

where

$$
\Gamma_q(z) \,=\, \prod_{n=0}^{\infty} \frac{1}{1\,-\,zq^n}
$$

The second formula also admits a recursion w.r.t. the rank N:

$$
{}^{q}\Psi_{\underline{z}_N}(\underline{p}_N) \,=\, \Gamma_q(q)^{N-2} \prod_{k=1}^{N-1} \oint \frac{dz_{N-1,k}}{2\pi\imath z_{N-1,k}} \Big(\frac{z_{N,1} z_{N,2}\,\cdot\,\ldots\,\cdot\, z_{NN}}{z_{N-1,1}\,\cdots\, z_{N-1,N-1}}\Big)^{p_{NN}}
$$

$$
\cdot {}^{\vee}\Delta(\underline{z}_{N-1})\ {}^{\vee}Q^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}}(\underline{z}_N, \underline{z}_{N-1}|p_{NN}|q)\ \cdot\ {}^{q}\Psi_{\underline{z}_{N-1}}(\underline{p}'_N)
$$

where

$$
{}^{\vee}\Delta(\underline{z}_{N-1}) \,=\, \prod_{k\neq j} \Gamma_q\Big(\frac{z_{N-1,\,k}}{z_{N-1,\,j}}\Big)
$$

and

$$
{}^{\vee}Q^{\mathfrak{gl}_N}_{\mathfrak{gl}_{N-1}}(\underline{z}_N, \underline{z}_{N-1}|p_{NN}|q) \,=\, \prod_{i=1}^{N}\prod_{k=1}^{N-1} \Gamma_q\Big(\frac{z_{N,\,i}}{z_{N-1,\,k}}\Big)
$$

- **Link to Riemann-Roch-Hirzebruch formula.**

$$\mathcal{M}_d(\mathbb{P}^1) := \Big\{\text{holomorphic maps } \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \text{ of degree } d\Big\}$$
$$= \Big\{ a_1(y_1,y_2),\ a_2(y_1,y_2) \,\Big|\, \deg a_i(\underline{y}) = d\Big\}\Big/\mathbb{C}^*$$
$$= \mathbb{P}^{2d+1}$$

There is an action of $G = S^1 \times U(2)$. Let $\mathcal{L}_k$ is such that $E_{ii}\cdot\mathcal{L}_k = k\mathcal{L}_k$; the G-equivariant Euler characteristic equals

$$\chi_G\big(\mathcal{L}_k(n)\big) = \sum_{k=0}^{N(d+1)-1} (-1)^k\, \mathrm{Tr}_{H^k(\mathbb{P}^{2d+1},\mathcal{L}_k(n))}\, e^{\hbar L_0+\lambda_1 E_{11}+\lambda_2 E_{22}}$$

$$= A^{(d)}_{n,\,k} = (z_1 z_2)^k \oint \frac{dt}{2\pi\imath\ t^{n+1}} \prod_{j=0}^{d} \frac{1}{(1-z_1 t q^j)\,(1-z_2 t q^j)}$$

where $q = e^{\hbar}$, and $z_i = e^{\lambda_i}$.

Proposition 1 :

$$\boxed{A^{(\infty)}_{n,\,k} := \lim_{d\to\infty} A^{(d)}_{n,k} = {}^{q}\Psi_{z_1,z_2}(n+k,\ k)}$$

For the equivariant cohomology

$$H^*_G\big(\mathbb{P}^{2d+1}\big) = \mathbb{C}[x,\hbar]\Big/\prod_{i=1}^{2}\prod_{j=0}^{d}(x-\lambda_i-\imath\hbar j)$$

the Riemann-Roch-Hirzebruch formula reads

$$\chi_G\big(\mathcal{L}_k(n)\big) = \big\langle \mathrm{Ch}\,_G(\mathcal{L}_k(n)),\ \mathrm{Td}_G(\mathcal{T}\mathbb{P}^{2d+1})\big\rangle$$

where

$$\mathrm{Ch}\,_G\big(\mathcal{L}_k(n)\big) = e^{nx+k(\lambda_1+\lambda_2)}$$

$$\mathrm{Td}_G(\mathcal{T}\mathbb{P}^{2d+1}) = \prod_{i=1}^{2}\prod_{j=0}^{d} \frac{x-\lambda_i-\imath\hbar j}{(1-e^{\lambda_i+\imath\hbar j-x})}$$

- **Connection to quantum K-theory.**

Let $L_+X = \mathrm{Map}(S^1,\, X)$; the algebraic version of $\widetilde{L_+\mathbb{P}^{N-1}}$ is

$$\mathcal{M}_d(\mathbb{P}^{N-1}) \;=\; \Big\{\, a_1(y_1,y_2)\,,\;\ldots,\; a_N(y_1,y_2)\Big\}\Big/\mathbb{C}^* \;=\; \mathbb{P}^{N(d+1)-1}$$

Conjecture 1 :

$${}^q\Psi_{\underline{z}}(n+k,\, k,\ldots,k) \;=\; (z_1\cdots z_N)^k\cdot\big\langle\, t^n,\; \big[\widetilde{L_+\mathbb{P}^{N-1}}\big]\,\big\rangle_{QK}$$

where the fundamental class of $\widetilde{L_+\mathbb{P}^{N-1}}$ *is given by*

$$\big[\widetilde{L_+\mathbb{P}^{N-1}}\big]_{QK} \;=\; \prod_{i=1}^{N}\Gamma_q\Big(\frac{z_i}{t}\Big)$$

- **Connection to quantum cohomology.**

When $q\longrightarrow 1$ the function ${}^q\Psi_{\underline{z}}(n+k,\, k,\ldots,k)$ tends to

$$\Phi_{\underline{\gamma}}(x) \;=\; \int d\gamma\; e^{-\imath\gamma x}\prod_{i=1}^{N}\Gamma\Big(\frac{\imath\gamma_i-\imath\gamma}{\hbar}\Big)$$

which is the solution of

$$\Big\{\,\prod_{i=1}^{N}\Big(\hbar\,\frac{\partial}{\partial x}\;-\;\imath\gamma_i\Big)\;-\;e^x\,\Big\}\cdot\Phi \;=\; 0$$

Conjecture 2 : *The fundamental class of* $\widetilde{L_+\mathbb{P}^{N-1}}$ *in* $QH^*\big(\widetilde{L_+\mathbb{P}^{N-1}}\big)\subset FH^*_{S^1}\big(\widetilde{L_+\mathbb{P}^{N-1}}\big)$ *is given by*

$$\big[\widetilde{L_+\mathbb{P}^{N-1}}\big]_{QH} \;=\; \prod_{i=1}^{N}\Gamma\Big(\frac{\imath\gamma_i-\imath\gamma}{\hbar}\Big)$$