The multicomponent 2D Toda hierarchy: string equations and dispersionless limits

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ENIGMA 2008
SISSA, Trieste, Italy, Octuber 2008
Integrable Systems, Geometry, Matrix Models and Applications

arXiv:0809.2720v1 [nlin.SI] 16 Sep 2008
arXiv:0810.2427v2 [math-ph] 15 Oct 2008

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Motivation Applications of multicomponent integrable systems:

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Methods Factorization problem in an infinite dimensional Lie group. Dressing and undressing

Results

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Extending results on multicomponent KP and dispersive Whitham hierarchies by Takasaki/Takebe (**MISGAM 2005**, SISSA, Trieste)

The Lie group

- Space of sequences $\{f(n)\}_{n\in\mathbb{Z}}$, $f(n)\in M_N(\mathbb{C})$. If $\mathbb{Z}\rightsquigarrow\mathbb{N}\Rightarrow$ we go from the infinite case to the semi-infinite case

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- Sequence $X_j:\mathbb{Z}\to M_N(\mathbb{C})$ acts by left multiplication

$$(X_j\Lambda^j)(f)(n) := X_j(n)\cdot f(n+j)$$

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- Lie subalgebras:

$$\mathfrak{g}_+ = \Big\{ \sum_{j \geqslant 0} X_j(n) \Lambda^j, \quad X_j(n) \in M_N(\mathbb{C}) \Big\}$$
$$\mathfrak{g}_- = \Big\{ \sum_{j < 0} X_j(n) \Lambda^j, \quad X_j(n) \in M_N(\mathbb{C}) \Big\}$$

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- Lie group G: invertible elements in $\mathfrak{g}$, Lie subgroups $G_\pm = G\cap\mathfrak{g}_\pm$

Gauss factorization

- Gauss factorization problem

$$g = g_-^{-1} \cdot g_+, \quad g_\pm \in G_\pm$$

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- G_+: set of invertible linear operators $\sum_{j \geqslant 0} g_j(n) \Lambda^j$
- G_-: set of invertible linear operators $1 + \sum_{j<0} g_j(n) \Lambda^j$

Vacuum dressing operators

$W_0, \bar{W}_0 \in G$, $s_a \in \mathbb{Z}$, $t_{ja}, \in \mathbb{C}$, Zero charge: $\sum_{a\in\mathcal{S}} s_a = 0$

$$W_0 := \sum_{k=1}^{N} E_{kk}\Lambda^{s_k}\exp\Big(\sum_{j=1}^{\infty} t_{jk}\Lambda^j\Big), \quad \bar{W}_0 := \sum_{k=1}^{N} E_{kk}\Lambda^{-s_{\bar{k}}}\exp\Big(\sum_{j=1}^{\infty} t_{j\bar{k}}\Lambda^{-j}\Big),$$

Three components

$$W_0 = \begin{pmatrix} \Lambda^{s_1}\,\mathrm{e}^{\sum_{j=1}^\infty t_{j1}\Lambda^j} & 0 & 0 \\ 0 & \Lambda^{s_2}\,\mathrm{e}^{\sum_{j=1}^\infty t_{j2}\Lambda^j} & 0 \\ 0 & 0 & \Lambda^{s_3}\,\mathrm{e}^{\sum_{j=1}^\infty t_{j3}\Lambda^j} \end{pmatrix}$$

$$\bar{W}_0 = \begin{pmatrix} \Lambda^{-s_{\bar{1}}}\,\mathrm{e}^{\sum_{j=1}^\infty t_{j\bar{1}}\Lambda^{-j}} & 0 & 0 \\ 0 & \Lambda^{-s_{\bar{2}}}\,\mathrm{e}^{\sum_{j=1}^\infty t_{j\bar{2}}\Lambda^{-j}} & 0 \\ 0 & 0 & \Lambda^{-s_{\bar{3}}}\,\mathrm{e}^{\sum_{j=1}^\infty t_{j\bar{3}}\Lambda^{-j}} \end{pmatrix}$$

Factorization problem

If g admits Gauss factorization, what about $W_0 \cdot g \cdot \bar{W}_0^{-1}$?

$$W_0 \cdot g \cdot \bar{W}_0^{-1} = S^{-1} \cdot \bar{S}, \quad \begin{aligned} S &= \mathbb{I}_N + \beta\Lambda^{-1} + \varphi_2\Lambda^{-2} + \cdots \in G_-, \\ \bar{S} &= e^{\phi} + \bar{\varphi}_1\Lambda + \bar{\varphi}_2\Lambda^2 + \cdots \in G_+ \end{aligned}$$

Dressing and Lax operators

After Ueno and Takasaki (1984)

Sato's dressing operators

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Lax operators:

$$L := W \cdot \Lambda \cdot W^{-1}, \qquad \bar{L} := \bar{W} \cdot \Lambda \cdot \bar{W}^{-1},$$

Notice the definition of $\bar{L}$

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New objects in the multicomponent context

$$C_{kl} := W \cdot E_{kl} \cdot W^{-1}, \qquad \bar{C}_{kl} := \bar{W} \cdot E_{kl} \cdot \bar{W}^{-1}$$

Λ-expansions

The Lax operators have the standard expressions

$$L = \Lambda + u_1(n) + u_2(n)\Lambda^{-1} + \cdots\,,$$
$$\bar{L}^{-1} = \bar{u}_0(n)\Lambda^{-1} + \bar{u}_1(n) + \bar{u}_2(n)\Lambda + \cdots\,,$$

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Multicomponent elements

$$C_{kl} = L^{s_k - s_l}\,\mathrm{e}^{\sum_{j=1}^{\infty}(t_{jk}-t_{jl})L^j}(E_{kl} + C_{kl,1}(n)\Lambda^{-1} + C_{kl,2}(n)\Lambda^{-2} + \cdots),$$
$$\bar{C}_{kl} = \bar{L}^{-s_{\bar{k}}+s_{\bar{l}}}\,\mathrm{e}^{\sum_{j=1}^{\infty}(t_{j\bar{k}}-t_{j\bar{l}})\bar{L}^{-j}}(\bar{C}_{kl,0}(n) + \bar{C}_{kl,1}(n)\Lambda + \bar{C}_{kl,2}(n)\Lambda^2 + \cdots).$$

Algebraic identities

$$\mathbb{I}_N = \sum_{k=1}^{N} C_{kk}, \quad \mathbb{I}_N = \sum_{k=1}^{N} \bar{C}_{kk},$$
$$C_{kl} C_{k'l'} = \delta_{lk'} C_{kl'}, \quad C_{kl} L = L C_{kl},$$
$$\bar{C}_{kl} \bar{C}_{k'l'} = \delta_{lk'} \bar{C}_{kl'}, \quad \bar{C}_{kl} \bar{L} = \bar{L} \bar{C}_{kl},$$

Definition

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2. The *zero-charge* shifts T_K for $K = (a, b)$ are defined as follows

$$s_a \to s_a + 1, \qquad\qquad s_b \to s_b - 1.$$

Lax equations

Theorem

- *Linear systems*

$$\partial_{ja} W = B_{ja} \cdot W, \qquad \partial_{ja} \bar{W} = B_{ja} \cdot \bar{W},$$
$$T_K W = \omega_K \cdot W, \qquad T_K \bar{W} = \omega_K \cdot \bar{W},$$

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$$\partial_{ja} W = B_{ja} \cdot W, \qquad \partial_{ja} \bar{W} = B_{ja} \cdot \bar{W},$$
$$T_K W = \omega_K \cdot W, \qquad T_K \bar{W} = \omega_K \cdot \bar{W},$$

- *Lax equations*

$$\begin{aligned}
\partial_{ja} L &= [B_{ja}, L], & \partial_{ja} \bar{L} &= [B_{ja}, \bar{L}],\\
\partial_{ja} C_{kk} &= [B_{ja}, C_{kk}], & \partial_{ja} \bar{C}_{kk} &= [B_{ja}, \bar{C}_{kk}],\\
T_K L &= \omega_K \cdot L \cdot \omega_K^{-1}, & T_K \bar{L} &= \omega_K \cdot \bar{L} \cdot \omega_K^{-1},\\
T_K C_{kk} &= \omega_K \cdot C_{kk} \cdot \omega_K^{-1}, & T_K \bar{C}_{kk} &= \omega_K \cdot \bar{C}_{kk} \cdot \omega_K^{-1},
\end{aligned}$$

Nonlinear partial differential-difference equations

$$
\begin{aligned}
\beta(n)E_{kk}-E_{kk}\beta(n+1)&=\partial_{1k}(\mathrm{e}^{\phi(n)})\,\mathrm{e}^{-\phi(n)}\\
\partial_{1\bar{k}}\beta(n)&=\mathrm{e}^{\phi(n)}\,E_{kk}\,\mathrm{e}^{-\phi(n-1)}\\
T_{(k,b)}\beta(n)E_{kk}-E_{kk}\beta(n+1)+\mathbb{I}_N-E_{kk}-\pi_b&=\mathrm{e}^{T_{(k,b)}\phi(n)}(\mathbb{I}_N-\bar{\pi}_b)\,\mathrm{e}^{-\phi(n)}\\
T_{(\bar{k},b)}\beta(n)(\mathbb{I}_N-\pi_b)-(\mathbb{I}_N-\pi_b)\beta(n)+\pi_b&=\mathrm{e}^{T_{(\bar{k},b)}\phi(n)}\,E_{kk}\,\mathrm{e}^{-\phi(n-1)}
\end{aligned}
$$

Nonlinear partial differential-difference equations

$$\beta(n)E_{kk}-E_{kk}\beta(n+1)=\partial_{1k}(\mathrm{e}^{\phi(n)})\,\mathrm{e}^{-\phi(n)}$$
$$\partial_{1\bar{k}}\beta(n)=\mathrm{e}^{\phi(n)}\,E_{kk}\,\mathrm{e}^{-\phi(n-1)}$$
$$T_{(k,b)}\beta(n)E_{kk}-E_{kk}\beta(n+1)+\mathbb{I}_N-E_{kk}-\pi_b=\mathrm{e}^{T_{(k,b)}\phi(n)}(\mathbb{I}_N-\bar{\pi}_b)\,\mathrm{e}^{-\phi(n)}$$
$$T_{(\bar{k},b)}\beta(n)(\mathbb{I}_N-\pi_b)-(\mathbb{I}_N-\pi_b)\beta(n)+\pi_b=\mathrm{e}^{T_{(\bar{k},b)}\phi(n)}\,E_{kk}\,\mathrm{e}^{-\phi(n-1)}$$

Matrix extension of the 2D Toda equation

$$\partial_{1\bar{k}'}\big(\partial_{1k}(\mathrm{e}^{\phi(n)})\cdot\mathrm{e}^{-\phi(n)}\big)=\mathrm{e}^{\phi(n)}\,E_{k'k'}\,\mathrm{e}^{-\phi(n-1)}\,E_{kk}-E_{kk}\,\mathrm{e}^{\phi(n+1)}\,E_{k'k'}\,\mathrm{e}^{-\phi(n)}$$

One component case

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1. 2D Toda equation

$$\partial_1 \partial_{\bar{1}}(\phi(n)) = \mathrm{e}^{\phi(n)-\phi(n-1)} - \mathrm{e}^{\phi(n+1)-\phi(n)}$$

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$$\partial_1 \partial_{\bar{1}}(\phi(n)) = e^{\phi(n)-\phi(n-1)} - e^{\phi(n+1)-\phi(n)}$$

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$$\partial_1 \partial_{\bar{1}}(\phi(n)) = \mathrm{e}^{\phi(n)-\phi(n-1)} - \mathrm{e}^{\phi(n+1)-\phi(n)}$$

2. Only the shift $T_{(s_1, s_{\bar{1}})} \leftrightsquigarrow n \to n+1$
3. If one sets the bared times to zero one obtains discrete KP

Discrete and discrete-continuous Toda type equations

Examples:

$$\Delta_{(\bar{k}',\bar{l})}\Big(\partial_{1k}(\mathrm{e}^{\phi(n)})\cdot\mathrm{e}^{-\phi(n)}\Big)=\mathrm{e}^{T_{(\bar{k}',\bar{l})}\phi(n)}\cdot E_{k'k'}\cdot\mathrm{e}^{-\phi(n-1)}E_{kk}-E_{kk}\,\mathrm{e}^{T_{(\bar{k}',\bar{l})}\phi(n+1)}\cdot E_{k'k'}\cdot\mathrm{e}^{-\phi(n)}$$

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$$\Delta_{(\bar{k}',\bar{l})}\big(\mathrm{e}^{T_{(k,b)}\phi(n)}\cdot(\mathbb{I}_N-\bar{\pi}_b)\cdot\mathrm{e}^{-\phi(n)}\big)=T_{(k,b)}\big(\mathrm{e}^{T_{(\bar{k}',\bar{l})}\phi(n)}\cdot E_{k'k'}\cdot\mathrm{e}^{-\phi(n-1)}\big)-E_{kk}\,T_{(k,b)}\big(\mathrm{e}^{T_{(\bar{k}',\bar{l})}\phi(n+1)}\cdot E_{k'k'}\cdot\mathrm{e}^{-\phi(n)}\big)$$

Introducing Orlov operators

Definition

The Orlov operators are defined as follows

$$M := WnW^{-1}, \qquad \bar{M} := \bar{W}n\bar{W}^{-1}$$

The Orlov operators satisfy

- $$[L, M] = L, \quad [L, C_{kk}] = 0, \quad [\bar{L}, \bar{M}] = \bar{L}, \quad [\bar{L}, \bar{C}_{kk}] = 0,$$

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- $$M = \mathcal{M} + \sum_{k=1}^{N} C_{kk}\big(s_k + \sum_{j=1}^{\infty} j t_{jk} L^j\big), \qquad \mathcal{M} = n + \mathfrak{g}_-$$
$$\bar{M} = \bar{\mathcal{M}} - \sum_{k=1}^{N} \bar{C}_{kk}\big(s_{\bar{k}} + \sum_{j=1}^{\infty} j t_{j\bar{k}} \bar{L}^{-j}\big), \quad \bar{\mathcal{M}} = n + \mathfrak{g}_+ \Lambda$$

Additional symmetries

Rewrite the factorization problem as

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with $g = h \cdot \bar{h}^{-1}$.

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with $g=h\cdot\bar h^{-1}$.

$$\partial_\mathfrak{b} h\cdot h^{-1}=\sum_{l,l'=1}^N F_{ll'}(n,\Lambda)E_{ll'},\quad \partial_\mathfrak{b}\bar h\cdot \bar h^{-1}=\sum_{l,l'=1}^N \bar F_{ll'}(n,\Lambda)E_{ll'}$$

$$T_\mathfrak{b} h\cdot h^{-1}=\sum_{l,l'=1}^N \mathcal{F}_{ll'}(n,\Lambda)E_{ll'},\quad T_\mathfrak{b}\bar h\cdot \bar h^{-1}=\sum_{l,l'=1}^N \bar{\mathcal{F}}_{ll'}(n,\Lambda)E_{ll'}$$

Here

$$F_{ll'}(n,\Lambda)=\sum_{i\in\mathbb{N},j\in\mathbb{Z}} F_{ll',ij}n^i\Lambda^j,\qquad F_{ll',ij}\in\mathbb{C}$$

Define

$$F := \sum_{l,l'=1}^{N} F_{ll'}(M, L)C_{ll'}, \quad \bar{F} := \sum_{l,l'=1}^{N} \bar{F}_{ll'}(\bar{M}, \bar{L})\bar{C}_{ll'}$$

$$\mathcal{F} := \sum_{l,l'=1}^{N} \mathcal{F}_{ll'}(M, L)C_{ll'}, \quad \bar{\mathcal{F}} := \sum_{l,l'=1}^{N} \bar{\mathcal{F}}_{ll'}(\bar{M}, \bar{L})\bar{C}_{ll'}$$

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$$H := F - \bar{F}, \qquad \mathcal{H} := \mathcal{F} \cdot \bar{\mathcal{F}}^{-1}$$

Theorem

1. *The Sato operators W and $\bar{W}$ satisfy*

$$\partial_{\mathfrak{b}} W = -H_- \cdot W, \qquad \partial_{\mathfrak{b}} \bar{W} = H_+ \cdot \bar{W},$$
$$T_{\mathfrak{b}} W = \mathcal{H}_- \cdot W, \qquad T_{\mathfrak{b}} \bar{W} = \mathcal{H}_+ \cdot \bar{W}$$

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$$T_{б} W = \mathcal{H}_- \cdot W, \qquad T_{б} \bar{W} = \mathcal{H}_+ \cdot \bar{W}$$

2. *The Lax and Orlov operators verify*

$$\partial_{б} L = [-H_-, L], \quad \partial_{б} M = [-H_-, M], \quad \partial_{б} C_{kk} = -[H_-, C_{kk}]$$
$$\partial_{б} \bar{L} = [H_+, \bar{L}], \quad \partial_{б} \bar{M} = [H_+, \bar{M}], \quad \partial_{б} \bar{C}_{kk} = [H_+, \bar{C}_{kk}]$$
$$T_{б} L = \mathcal{H}_- L \mathcal{H}_-^{-1}, \quad T_{б} M = \mathcal{H}_- M \mathcal{H}_-^{-1} \quad T_{б} C_{kk} = \mathcal{H}_- C_{kk} \mathcal{H}_-^{-1}$$
$$T_{б} \bar{L} = \mathcal{H}_+ \bar{L} \mathcal{H}_+^{-1} \quad T_{б} \bar{M} = \mathcal{H}_+ \bar{M} \mathcal{H}_+^{-1} \quad T_{б} \bar{C}_{kk} = \mathcal{H}_+ \bar{C}_{kk} \mathcal{H}_+^{-1}$$

String equations. Block Toeplitz/Hankel operators

Suppose that the initial condition g satisfies

$$\boxed{g\bar{F}_0 = F_0 g}, \quad F_0 := \sum_{l,l'=1}^{N} F_{ll'}(n,\Lambda)E_{ll'}, \quad \bar{F}_0 = \sum_{l,l'=1}^{N} \bar{F}_{ll'}(n,\Lambda)E_{ll'},$$

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$F(M,L) := \sum_{l,l'=1}^{N} F_{ll'}(M,L)C_{ll'}$, $\bar{F}(\bar{M},\bar{L}) = \sum_{l,l'=1}^{N} \bar{F}_{ll'}(\bar{M},\bar{L})\bar{C}_{ll'}$
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String equation

$$\boxed{F(M,L) = \bar{F}(\bar{M},\bar{L})}$$

This is equivalent to invariance under an additional symmetry

For $\{\ell_a\}_{a\in\mathcal{S}} \subset \mathbb{Z}$

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$$g\cdot\Big(\sum_{k=1}^{N}E_{kk}\Lambda^{-\ell_{\bar{k}}}\Big)=\Big(\sum_{k=1}^{N}E_{kk}\Lambda^{\ell_k}\Big)\cdot g\Leftrightarrow\sum_{k=1}^{N}C_{kk}L^{\ell_k j}=\sum_{k=1}^{N}\bar{C}_{kk}\bar{L}^{-\ell_{\bar{k}}j}$$

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If $\mathbb{S}_\pm:=\{a\in\mathbb{S}:\pm\ell_a>0\}$, $\mathbb{S}_0:=\{a\in\mathbb{S}:\ell_a=0\}$,
$\bar{\mathbb{S}}_\pm:=\{a\in\bar{\mathbb{S}}:\pm\ell_a>0\}$, $\bar{\mathbb{S}}_0:=\{a\in\bar{\mathbb{S}}:\ell_a=0\}$

For $\{\ell_a\}_{a\in\mathcal{S}}\subset\mathbb{Z}$

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$$\boxed{\begin{aligned}\Big(\sum_{a\in\mathbb{S}_+\cup\mathbb{S}_0\cup\bar{\mathbb{S}}_+}\partial_{j\ell_a,a}\Big)(L)=\Big(\sum_{a\in\mathbb{S}_+\cup\mathbb{S}_0\cup\bar{\mathbb{S}}_+}\partial_{j\ell_a,a}\Big)(\bar{L})=0,\\ \Big(\sum_{a\in\mathbb{S}_-\cup\mathbb{S}_0\cup\bar{\mathbb{S}}_-}\partial_{j|\ell_a|,a}\Big)(L)=\Big(\sum_{a\in\mathbb{S}_-\cup\mathbb{S}_0\cup\bar{\mathbb{S}}_-}\partial_{j|\ell_a|,a}\Big)(\bar{L})=0,\end{aligned}}$$

When

$$\sum_{a\in\mathcal{S}}\ell_a = 0;$$

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$$\sum_{a\in\mathcal{S}} \ell_a = 0;$$

the **Lax operators are periodic**

$$L(s_1+\ell_1,\dots,s_N+\ell_N,s_{\bar 1}+\ell_{\bar 1},\dots,s_{\bar N}+\ell_{\bar N}) = L(s_1,\dots,s_N,s_{\bar 1},\dots,s_{\bar N})$$
$$\bar L(s_1+\ell_1,\dots,s_N+\ell_N,s_{\bar 1}+\ell_{\bar 1},\dots,s_{\bar N}+\ell_{\bar N}) = \bar L(s_1,\dots,s_N,s_{\bar 1},\dots,s_{\bar N})$$

If $g=\sum_{k_1,k_2=1}^{N} g_{k_1 k_2} E_{k_1 k_2}$, $g_{k_1 k_2}=\sum_{j\in\mathbb{Z}} g_{j,k_1 k_2}(n)\Lambda^j$

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- For $\ell_{k_1}=0$ with $\ell_{\bar{k}_2}\neq 0$ we have a diagonal band structure being $|\ell_{\bar{k}_2}|$ its width, and for $\ell_{\bar{k}_2}=0$ with $\ell_{k_1}\neq 0$ a $|\ell_{k_1}|\times|\ell_{k_1}|$ block bi-infinite matrix.

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Given a block matrix $\Omega=(\Omega_{i,j})_{i,j\in\mathbb{Z}\text{ or }\mathbb{N}}$ made up with $p\times q$-blocks $\Omega_{i,j}$ we say that Ω is a block Toeplitz matrix if $\Omega_{i+1,j+1}=\Omega_{i,j}$ and a block Hankel matrix if $\Omega_{i+1,j-1}=\Omega_{i,j}$

Baker functions

After Adler and van Moerbeke

1. For any complex number $z \in \mathbb{C}$

$$\chi(z) := \{z^n \mathbb{I}_N\}_{n \in \mathbb{Z}}$$

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3. Baker functions

$$\psi = W \cdot \chi, \quad \bar{\psi} = \bar{W} \cdot \chi,$$

$$\psi = z^n \varphi(z,n)\psi_0(z), \qquad \varphi := \mathbb{I}_N + \varphi_1(n)z^{-1} + \varphi_2(n)z^{-2} + \cdots$$

$$\psi_0 := \sum_{k=1}^{N} E_{kk} z^{s_k}\, \mathrm{e}^{\sum_{j=1}^{\infty} t_{jk} z^j},$$

$$\bar\psi = z^n \bar\varphi(z,n)\bar\psi_0(z), \qquad \bar\varphi := \bar\varphi_0(n) + \bar\varphi_1(n)z + \bar\varphi_2(n)z^2 + \cdots$$

$$\bar\psi_0 := \sum_{k=1}^{N} E_{kk} z^{-s_{\bar k}}\, \mathrm{e}^{\sum_{j=1}^{\infty} t_{j\bar k} z^{-j}}$$

Given *diagonal* operators of the form

$$F := \sum_{k=1}^{N} F_k C_{kk}, \qquad F_k := \sum_{i \geqslant 0, j \in \mathbb{Z}} F_{kij} M^i L^j,$$

$$\bar{F} := \sum_{k=1}^{N} F_{\bar{k}} \bar{C}_{kk}, \qquad F_{\bar{k}} := \sum_{i \geqslant 0, j \in \mathbb{Z}} F_{\bar{k}ij} \bar{M}^i \bar{L}^j,$$

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we can write

$$F(\psi) = (\psi) \sum_{k=1}^{N} \overleftarrow{F_k\left(z\frac{\mathrm{d}}{\mathrm{d}z}, z\right)} E_{kk} = \sum_{i\geqslant 0, j\in\mathbb{Z}} F_{kij} z^j \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^i (\psi),$$

$$\bar{F}(\bar{\psi}) = (\bar{\psi}) \sum_{k=1}^{N} \overleftarrow{F_{\bar{k}}\left(z\frac{\mathrm{d}}{\mathrm{d}z}, z\right)} E_{kk} = \sum_{i\geqslant 0, j\in\mathbb{Z}} \bar{F}_{kij} z^j \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^i (\bar{\psi})$$

Charge preserving Lax and Orlov–Schulman operators

1. Two sets of charge preserving Lax and Orlov–Schulman operators, $\{\mathscr{L}_a, \mathscr{M}_a\}_{a\in\mathcal{S}}$ and $\{\bar{\mathscr{L}}_a, \bar{\mathscr{M}}_a\}_{a\in\mathcal{S}}$.

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2. First family depends on two arbitrary given elements $I \in \mathbb{S}$ and $a' \in \mathcal{S}$, while the second set depends on two arbitrary elements $\bar{I} \in \bar{\mathbb{S}}$ and $a' \in \mathcal{S}$.

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3. These sets of operators will lead to two dispersionless hierarchies, the first one is the genus 0 universal Whitham hierarchy while the second provides a multicomponent version of the dispersionless 2D Toda hierarchy as presented by Takasaki and Takebe

Definition

$$\mathcal{T}_a := \begin{cases} T_{(l,a')}, & a = l, \\ T_{(a,l)}, & a \neq l \end{cases} \quad \bar{\mathcal{T}}_a := \begin{cases} T_{(\bar{l},a')}, & a = \bar{l}, \\ T_{(a,\bar{l})}, & a \neq \bar{l}, \end{cases}$$

Definition

$$\mathscr{K}_a := \begin{cases} 1 + \beta_{ll}\mathcal{T}_l^{-1} + \varphi_{2,ll}\mathcal{T}_l^{-2} + \cdots, & a = l \\ \beta_{lk} + \varphi_{2,lk}\mathcal{T}_k^{-1} + \cdots, & a = k \neq l, \\ \bar{\varphi}_{0,lk} + \bar{\varphi}_{1,lk}\mathcal{T}_{\bar{k}}^{-1} + \cdots, & a = \bar{k}, \end{cases}$$

$$\bar{\mathscr{K}}_a := \begin{cases} 1 + \beta_{ll}\bar{\mathcal{T}}_l^{-1} + \varphi_{2,ll}\bar{\mathcal{T}}_l^{-2} + \cdots, & a = l, \\ \beta_{lk} + \varphi_{2,lk}\bar{\mathcal{T}}_k^{-1} + \cdots, & a = k \neq l \\ \bar{\varphi}_{0,lk} + \bar{\varphi}_{1,lk}\bar{\mathcal{T}}_{\bar{k}}^{-1} + \cdots, & a = \bar{k}, \end{cases}$$

Definition

Charge preserving Sato operators

$$\mathscr{W}_a := \mathscr{K}_a \circ \mathscr{W}_{0,a}, \qquad \mathscr{W}_{0,a} := \exp(\mathscr{T}_a), \qquad \mathscr{T}_a := \sum_{j=1}^{\infty} t_{ja} \mathcal{T}_a^j,$$

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Charge preserving Lax operators

$$\mathscr{L}_a := \mathscr{W}_a \circ \mathcal{T}_a \circ \mathscr{W}_a^{-1}, \qquad \bar{\mathscr{L}}_a := \bar{\mathscr{W}}_a \circ \bar{\mathcal{T}}_a \circ \bar{\mathscr{W}}_a^{-1}$$

Definition

Charge preserving Orlov operators

$$\mathscr{M}_a := n - n_{0,a} + \operatorname{sg}(a)\mathscr{W}_a \circ s_a \circ \mathscr{W}_a^{-1}, \quad \operatorname{sg}(a) := \begin{cases} 1, & a \in \mathbb{S}, \\ -1, & a \in \bar{\mathbb{S}}. \end{cases}$$

$$\bar{\mathscr{M}}_a := n - n_{0,a} + \operatorname{sg}(a)\bar{\mathscr{W}}_a \circ s_a \circ \bar{\mathscr{W}}_a^{-1}, \qquad n_{0,a} := \begin{cases} 1, & a \in \mathbb{S} - \{l\}, \\ 0, & a \notin \mathbb{S} - \{l\}, \end{cases}$$

Weak identities

We present some identities between the charge preserving operators and standard operators that hold in a weak sense, i.e. when we act on the l-th row of Sato operators or Baker functions

Theorem

$$
\begin{aligned}
&[F(\mathscr{M}_k,\mathscr{L}_k)](W_{lk})E_{lk}=[F(\bar{\mathscr{M}}_k,\bar{\mathscr{L}}_k)](W_{lk})E_{lk}=E_{ll}F(M,L)C_{kk}W,\\
&[\bar F(\mathscr{M}_{\bar k},\mathscr{L}_{\bar k})](\bar W_{lk})E_{lk}=[\bar F(\bar{\mathscr{M}}_{\bar k},\bar{\mathscr{L}}_{\bar k})](\bar W_{lk})E_{lk}=E_{ll}\bar F(\bar M,\bar L^{-1})\bar C_{kk}\bar W
\end{aligned}
$$

Theorem

$$[F(\mathscr{M}_k,\mathscr{L}_k)](W_{lk})E_{lk}=[F(\bar{\mathscr{M}}_k,\bar{\mathscr{L}}_k)](W_{lk})E_{lk}=E_{ll}F(M,L)C_{kk}W,$$
$$[\bar{F}(\mathscr{M}_{\bar{k}},\mathscr{L}_{\bar{k}})](\bar{W}_{lk})E_{lk}=[\bar{F}(\bar{\mathscr{M}}_{\bar{k}},\bar{\mathscr{L}}_{\bar{k}})](\bar{W}_{lk})E_{lk}=E_{ll}\bar{F}(\bar{M},\bar{L}^{-1})\bar{C}_{kk}\bar{W}$$

$$[F(\mathscr{M}_k,\mathscr{L}_k)](\psi_{lk})=[F(\bar{\mathscr{M}}_k,\bar{\mathscr{L}}_k)](\psi_{lk})=(\psi_{lk})\overleftarrow{F\Big(z\frac{\mathrm{d}}{\mathrm{d}z},z\Big)},$$
$$[\bar{F}(\mathscr{M}_{\bar{k}},\mathscr{L}_{\bar{k}})](\bar{\psi}_{lk})=[\bar{F}(\bar{\mathscr{M}}_{\bar{k}},\bar{\mathscr{L}}_{\bar{k}})](\bar{\psi}_{lk})=(\bar{\psi}_{lk})\overleftarrow{\bar{F}\Big(z\frac{\mathrm{d}}{\mathrm{d}z},z^{-1}\Big)}$$

We now introduce algebras of charge preserving operators generated by charge preserving shifts operators

Definition

$$\mathfrak{t}_a := \Big\{ \sum_{j\in\mathbb{Z}} c_j \mathcal{T}_a^j \Big\}, \qquad \bar{\mathfrak{t}}_a := \Big\{ \sum_{j\in\mathbb{Z}} c_j \bar{\mathcal{T}}_a^j \Big\}$$

$$\begin{cases} \mathfrak{t}_{a,>} := \Big\{\sum_{j>0} c_j \mathcal{T}_a^j\Big\} & \mathfrak{t}_{a,\leqslant} := \Big\{\sum_{j\leqslant 0} c_j \mathcal{T}_a^j\Big\}, \quad a \neq l \\ \mathfrak{t}_{l,\geqslant} := \Big\{\sum_{j\geqslant 0} c_j \mathcal{T}_l^j\Big\} & \mathfrak{t}_{l,<} := \Big\{\sum_{j<0} c_j \mathcal{T}_l^j\Big\} \end{cases}$$

$$\begin{cases} \bar{\mathfrak{t}}_{a,>} := \Big\{\sum_{j>0} c_j (\bar{\mathcal{T}}_a^j - 1)\Big\}, & \bar{\mathfrak{t}}_{a,<} := \Big\{\sum_{j<0} c_j (\bar{\mathcal{T}}_a^j - 1)\Big\}, \quad a \neq l, \bar{l} \\ \bar{\mathfrak{t}}_{l,\geqslant} := \Big\{\sum_{j\geqslant 0} c_j \bar{\mathcal{T}}_l^j\Big\}, & \bar{\mathfrak{t}}_{l,<} := \Big\{\sum_{j<0} c_j \bar{\mathcal{T}}_l^j\Big\}, \\ \bar{\mathfrak{t}}_{\bar{l},>} := \Big\{\sum_{j>0} c_j (\bar{\mathcal{T}}_{\bar{l}}^j - 1)\Big\} & \bar{\mathfrak{t}}_{\bar{l},<} := \Big\{\sum_{j<0} c_j (\bar{\mathcal{T}}_{\bar{l}}^j - 1)\Big\}, \quad a' \neq l, \\ \bar{\mathfrak{t}}_{\bar{l},>} := \Big\{\sum_{j>0} c_j \bar{\mathcal{T}}_{\bar{l}}^j\Big\} & \bar{\mathfrak{t}}_{\bar{l},\leqslant} := \Big\{\sum_{j\leqslant 0} c_j \bar{\mathcal{T}}_{\bar{l}}^j\Big\}, \quad a' = l. \end{cases}$$

Red color in the previous slide will be denoted by $+$

$$\begin{cases} F(\mathscr{M}_k,\mathscr{L}_k)_+(E_{ll}W)=F(\bar{\mathscr{M}}_k,\bar{\mathscr{L}}_k)_+(E_{ll}W)=E_{ll}(F(M,L)C_{kk})_+W\\ F(\mathscr{M}_k,\mathscr{L}_k)_+(E_{ll}\bar{W})=F(\bar{\mathscr{M}}_k,\bar{\mathscr{L}}_k)_+(E_{ll}\bar{W})=E_{ll}(F(M,L)C_{kk})_+\bar{W}\end{cases}$$

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Definition

$$\mathscr{B}_{ja} := (\mathscr{L}_a^j)_+, \qquad \bar{\mathscr{B}}_{ja} := (\bar{\mathscr{L}}_a^j)_+$$

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Theorem

$$\partial_{ja}(E_{ll}W) = \mathscr{B}_{ja}(E_{ll}W) = \bar{\mathscr{B}}_{ja}(E_{ll}W),$$
$$\partial_{ja}(E_{ll}\bar{W}) = \mathscr{B}_{ja}(E_{ll}\bar{W}) = \bar{\mathscr{B}}_{ja}(E_{ll}\bar{W}),$$

If the string equation

$$\sum_{k=1}^{N} F_k(M,L)C_{kk} = \sum_{k=1}^{N} F_{\bar{k}}(\bar{M},\bar{L}^{-1})\bar{C}_{kk}$$

holds, then

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holds, then

$$F_k(\mathscr{M}_k,\mathscr{L}_k)(\psi_{lk})=\Big(\sum_{a'\in\mathcal{S}} F_{a'}(\mathscr{M}_{a'},\mathscr{L}_{a'})_+\Big)(\psi_{lk})$$

$$F_k(\bar{\mathscr{M}}_k,\bar{\mathscr{L}}_k)(\psi_{lk})=\Big(\sum_{a'\in\mathcal{S}} F_{a'}(\bar{\mathscr{M}}_{a'},\bar{\mathscr{L}}_{a'})_+\Big)(\psi_{lk}),$$

$$F_{\bar{k}}(\mathscr{M}_{\bar{k}},\mathscr{L}_{\bar{k}})(\bar{\psi}_{lk})=\Big(\sum_{a'\in\mathcal{S}} F_{a'}(\mathscr{M}_{a'},\mathscr{L}_{a'})_+\Big)(\bar{\psi}_{lk}),$$

$$F_{\bar{k}}(\bar{\mathscr{M}}_{\bar{k}},\bar{\mathscr{L}}_{\bar{k}})(\psi_{lk})=\Big(\sum_{a'\in\mathcal{S}} F_{a'}(\bar{\mathscr{M}}_{a'},\bar{\mathscr{L}}_{a'})_+\Big)(\bar{\psi}_{lk}),$$

Dispersionless limit

$$\partial_{ja}(E_{ll}\psi) = \mathscr{B}_{ja}(E_{ll}\psi), \quad \partial_{ja}(E_{ll}\bar{\psi}) = \mathscr{B}_{ja}(E_{ll}\bar{\psi}) \quad a \in \mathcal{S}, j = 1, 2, \dots .$$

Dispersionless limit

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Slow variables

$$t_{\mathrm{sl},ja} = \epsilon t_{ja}, \qquad s_{\mathrm{sl},a} = \epsilon s_a, \qquad n_{\mathrm{sl}} = \epsilon n$$

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$$\partial_{ja}(E_{ll}\psi) = \mathscr{B}_{ja}(E_{ll}\psi), \quad \partial_{ja}(E_{ll}\bar{\psi}) = \mathscr{B}_{ja}(E_{ll}\bar{\psi}) \quad a \in \mathcal{S}, j = 1, 2, \dots .$$

Slow variables

$$t_{\text{sl},ja} = \epsilon t_{ja}, \qquad s_{\text{sl},a} = \epsilon s_a, \qquad n_{\text{sl}} = \epsilon n$$

$$\psi_a = \exp\left(\frac{\mathscr{S}_a}{\epsilon}\right), \qquad \mathscr{S}_a = \mathscr{S}_{a,0} + \epsilon \mathscr{S}_{a,1} + \cdots .$$

New variables

$$\sigma_{b,a} := s_{\mathrm{sl},a}, \qquad a \neq b, \qquad s_{\mathrm{sl},a} = \sigma_a, \qquad a \neq b,$$
$$\sigma_{b,b} := \sum_{a\in\mathcal{S}} s_{\mathrm{sl},a}, \qquad s_{\mathrm{sl},b} = \sigma_b - \sum_{a\neq b}\sigma_a,$$

and observe that

$$\frac{\partial}{\partial\sigma_{b,a}} = \frac{\partial}{\partial s_{\mathrm{sl},a}} - \frac{\partial}{\partial s_{\mathrm{sl},b}}, \qquad a \neq b.$$

$$\partial_a := \begin{cases} \dfrac{\partial}{\partial\sigma_{l,a}}, & a \neq l, \\ -\dfrac{\partial}{\partial\sigma_{l,a'}}, & a = l, \end{cases}, \qquad \bar\partial_a := \begin{cases} \dfrac{\partial}{\partial\sigma_{\bar l,a}}, & a \neq \bar l, \\ -\dfrac{\partial}{\partial\sigma_{\bar l,a'}}, & a = \bar l. \end{cases}$$

Theorem

In the limit $\epsilon \longrightarrow 0$ we have that

$$\mathcal{T}_a^j(\exp(\mathscr{S}_b/\epsilon)) = \exp(j\partial_a(\mathscr{S}_{b,0}) + O(\epsilon))\exp(\mathscr{S}_b/\epsilon)$$
$$\bar{\mathcal{T}}_a^j(\exp(\mathscr{S}_b/\epsilon)) = \exp(j\bar{\partial}_a(\mathscr{S}_{b,0}) + O(\epsilon))\exp(\mathscr{S}_b/\epsilon)$$
$$\partial_{ja}(\exp(\mathscr{S}_b/\epsilon)) = (\partial_{\mathrm{sl},ja}(\mathscr{S}_{b,0}) + O(\epsilon))\exp(\mathscr{S}_b/\epsilon).$$

Dispersionless limits of the shift operators $\mathscr{B}_{ja}, \bar{\mathscr{B}}_{ja}$

$$
\begin{aligned}
\mathscr{P}_{jl} &= Y^j + \mathscr{B}_{jl,j-1|0} Y^{j-1} + \cdots + \mathscr{B}_{jl,0|0}, \quad a \neq l \\
\mathscr{P}_{ja} &= \mathscr{B}_{ja,j|0} Y^j + \cdots + \mathscr{B}_{ja,1|0} Y \\
\bar{\mathscr{P}}_{j\bar{l}} &= \bar{\mathscr{B}}_{j\bar{l},j|0} Y^j + \mathscr{B}_{j\bar{l},j-1|0} Y^{j-1} + \cdots + \bar{\mathscr{B}}_{j\bar{l},1|0} Y - (1-\delta_{la'}) \sum_{i=1}^{j} \bar{\mathscr{B}}_{j\bar{l},i|0} \\
\bar{\mathscr{P}}_{ja} &= \bar{\mathscr{B}}_{ja,j|0} Y^j + \cdots + \bar{\mathscr{B}}_{ja,1|0} Y - (1-\delta_{al}) \sum_{i=1}^{j} \bar{\mathscr{B}}_{ja,i|0}, \quad a \neq \bar{l}
\end{aligned}
$$

where

$$
\mathscr{B}_{ja,i|0} := \lim_{\epsilon \to 0} \mathscr{B}_{ja,i}, \qquad \bar{\mathscr{B}}_{ja,i|0} := \lim_{\epsilon \to 0} \bar{\mathscr{B}}_{ja,i}
$$

Hamilton–Jacobi equations

Two families of Hamilton–Jacobi type equations in the dispersionless limit

$$\partial_{\mathrm{sl},ja}(\mathscr{S}_{b,0}) = \mathscr{P}_{ja}\Big(\mathrm{e}^{\partial_a \mathscr{S}_{b,0}} \Big), \qquad \text{Whitham type}$$

$$\partial_{\mathrm{sl},ja}(\mathscr{S}_{b,0}) = \bar{\mathscr{P}}_{ja}\Big(\mathrm{e}^{\partial_a \mathscr{S}_{b,0}} \Big) \qquad \text{dToda type}$$

Let us consider the equation

$$\partial_x(\mathscr{S}_{b,0}) = \exp(\partial_l \mathscr{S}_{b,0}) + q_{a'} = \exp(-\partial_{a'} \mathscr{S}_{b,0}) + q_{a'}, \quad q_{a'} := \partial_{a'}(\beta_{l|l|-1}),$$

Inserting the substitution $a' \rightsquigarrow a$ we get

$$\partial_a(\mathscr{S}_{b,0}) = -\log(\partial_x(\mathscr{S}_{b,0}) - q_a) \qquad a \neq l.$$

Definition

Let us introduce the dispersionless Lax function $z_a = z_a(\mathbf{s}_{\mathrm{sl}}, \mathbf{t}_{\mathrm{sl}})$ by the implicit relations

$$p = \partial_x \mathscr{S}_{a,0}(z_a)$$

and the dispersionless Orlov–Schulman function by
$$m_a := \frac{\partial \mathscr{S}_{a,0}}{\partial z}\bigg|_{z=z_a}$$

Therefore for $a \neq l$ we have

$$\begin{aligned}
(\partial_a(\mathscr{S}_{b,0}))\Big|_{z=z_b} &= -\log(p-q_a), && a \neq l\\
(\partial_{\mathrm{sl},ja}(\mathscr{S}_{b,0}))\Big|_{z=z_b} &= \mathscr{P}_{ja}\Big(\frac{1}{p-q_a}\Big) =: \Omega_{ja}, && a \neq l\\
(\partial_{\mathrm{sl},jl}(\mathscr{S}_{b,0}))\Big|_{z=z_b} &= \mathscr{P}_{jl}(p-q_{a'}) =: \Omega_{jl}, && j>1
\end{aligned}$$

Therefore for $a \neq l$ we have

$$
\begin{aligned}
\left(\partial_a(\mathscr{S}_{b,0})\right)\Big|_{z=z_b} &= -\log(p - q_a), && a \neq l\\
\left(\partial_{\mathrm{sl},ja}(\mathscr{S}_{b,0})\right)\Big|_{z=z_b} &= \mathscr{P}_{ja}\Big(\frac{1}{p - q_a}\Big) =: \Omega_{ja}, && a \neq l\\
\left(\partial_{\mathrm{sl},jl}(\mathscr{S}_{b,0})\right)\Big|_{z=z_b} &= \mathscr{P}_{jl}(p - q_{a'}) =: \Omega_{jl}, && j > 1
\end{aligned}
$$

Given two functions $f = f(p,x)$ and $g = g(p,x)$ of p and x we define the standard Poisson bracket

$$
\{f,g\}_0 := \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial x}
$$

As shows the following Theorem the functions (z_b, m_b) satisfy the universal genus 0 Whitham hierarchy

Theorem

The functions (z_b, m_b) verify

$$\{z_b, m_b\}_0 = 1,$$

and the evolution equations

$$\begin{aligned}\partial_a z_b &= \{-\log(p-q_a), z_b\}_0, & \partial_a m_b &= \{-\log(p-q_a), m_b\}_0, \quad a \neq l.\\ \partial_{\mathrm{sl},ja} z_b &= \{\Omega_{ja}, z_b\}_0, & \partial_{\mathrm{sl},ja} m_b &= \{\Omega_{ja}, m_b\}_0.\end{aligned}$$

Introduce a variable X such that $\partial_X := \partial_l = -\partial_{\bar{l}}$, it comes from a discrete s type variable while x comes from a continuous t variable

Introduce a variable X such that $\partial_X := \partial_l = -\partial_{\bar{l}}$, it comes from a discrete s type variable while x comes from a continuous t variable

Definition

The dispersionless Lax function $\bar{z}_a$ is defined by the implicit relation

$$\partial_X \mathscr{S}_{a,0}|_{z=\bar{z}_a} = \log P,$$

while the dispersionless Orlov–Schulman operator is defined by

$$\bar{m}_a := \bar{z}_a \frac{\partial \mathscr{S}_{a,0}}{\partial z}\bigg|_{z=\bar{z}_a}.$$

multicomponent dispersionless 2D Toda for $\mathscr{S}$

$$\partial_X(\mathscr{S}_{b,0})\Big|_{z=\bar{z}_b} = \log P,$$

$$\partial_a(\mathscr{S}_{b,0})\Big|_{z=\bar{z}_b} = \log P_a, \qquad P_a := \frac{1}{1+\rho_a P^{-1}} \quad a \neq l, \bar{l},$$

$$(\partial_{\mathrm{sl},ja}(\mathscr{S}_{b,0}))\Big|_{z=\bar{z}_b} = \bar{\mathscr{P}}_{ja}(P_a) =: \bar{\Omega}_{ja}, \qquad a \neq l, \bar{l},$$

$$\partial_{\mathrm{sl},j\bar{l}}(\mathscr{S}_{b,0})\Big|_{z=\bar{z}_b} = \bar{\mathscr{P}}_{j\bar{l}}(P^{-1}) =: \bar{\Omega}_{j\bar{l}},$$

$$(\partial_{\mathrm{sl},jl}(\mathscr{S}_{b,0}))\Big|_{z=\bar{z}_b} = \bar{\mathscr{P}}_{jl}(P) =: \bar{\Omega}_{jl}, \qquad j > 1.$$

Given two functions $f = f(p, x)$ and $g = g(p, x)$ of p and x we define the modified Poisson bracket

$$\{f, g\}_1 := p\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - p\frac{\partial g}{\partial p}\frac{\partial f}{\partial x}.$$

Next we show that this dispersionless limit represents a multicomponent generalization of the standard dispersionless 2D Toda hierarchy. This later corresponds to the flows associated with $t_{\mathrm{sl},jI}, t_{\mathrm{sl},j\bar{I}}, j = 1, 2, \dots$.

Next we show that this dispersionless limit represents a multicomponent generalization of the standard dispersionless 2D Toda hierarchy. This later corresponds to the flows associated with $t_{\mathrm{sl},jl}, t_{\mathrm{sl},j\bar{l}}$, $j = 1, 2, \dots$.

Theorem

The relation

$$\{\bar{z}_b, \bar{m}_b\}_1 = \bar{z}_b$$

and the following evolution equations are verified

$$\begin{aligned} \partial_a \bar{z}_b &= \{\log(P_a), \bar{z}_b\}_1, & \partial_a \bar{m}_b &= \{\log(P_a), \bar{m}_b\}_1, \quad a \neq l, \bar{l} \\ \partial_{ja} \bar{z}_b &= \{\bar{\Omega}_{ja}, \bar{z}_b\}_1, & \partial_{ja} \bar{m}_b &= \{\bar{\Omega}_{ja}, \bar{m}_b\}_1. \end{aligned}$$

Theorem

If the string equations

$$\sum_{k=1}^{N} F_k(M, L)C_{kk} = \sum_{k=1}^{N} F_{\bar{k}}(\bar{M}, \bar{L}^{-1})\bar{C}_{kk}$$

hold, their corresponding dispersionless limits

$$F_{a,0}(z_a, m_a) = \sum_{b\in\mathcal{S}} F_{b,0+}, \quad \bar{F}_{a,0}(\bar{z}_a, \bar{m}_a) = \sum_{b\in\mathcal{S}} \bar{F}_{b,0+}, \quad \forall a \in \mathcal{S}$$

are satisfied