Semiclassical Limit of focusing NLS

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Outline

- Introduction
- Semiclassical limit of the focusing NLS equation before the time of gradient catastrophe
- Painlevé I near the gradient catastrophe
- *tritronquées* solution to Painlevé I
- Numerical methods
- Outlook

Nonlinear Schrödinger equation

- NLS equation

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} - \rho|\psi|^2\psi = 0$$

  $\rho = -1$: focusing, $\rho = 1$: defocusing

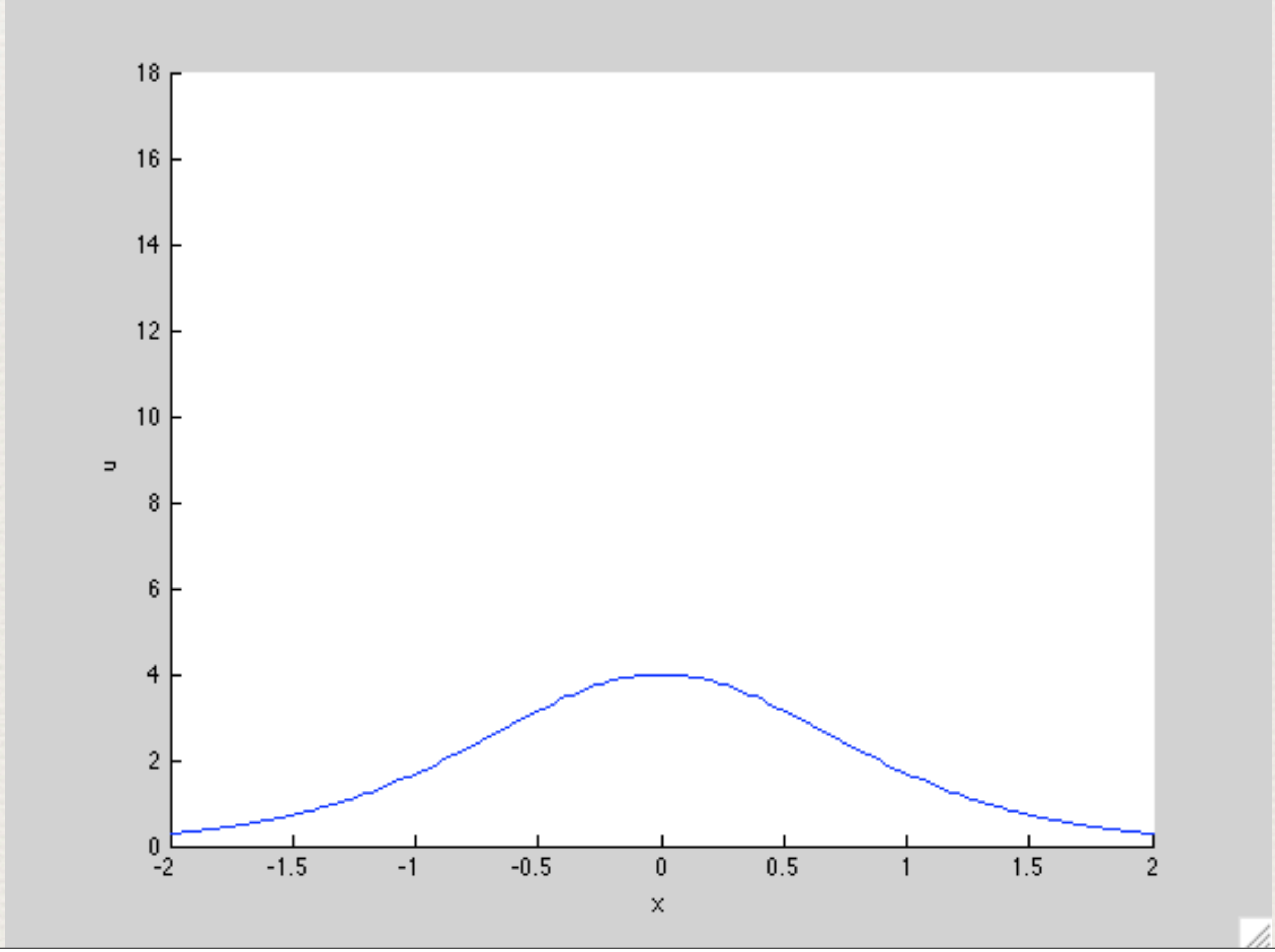
- applications in nonlinear optics (fiber optics), hydrodynamics,...

- completely integrable (Zakharov, Shabat)

Breather

$$\psi = 4\exp(it/2)\frac{\cosh(3x)+3\exp(4it)\cosh(x)}{\cosh(4x)+4\cosh(2x)+3\cos(4t)}$$

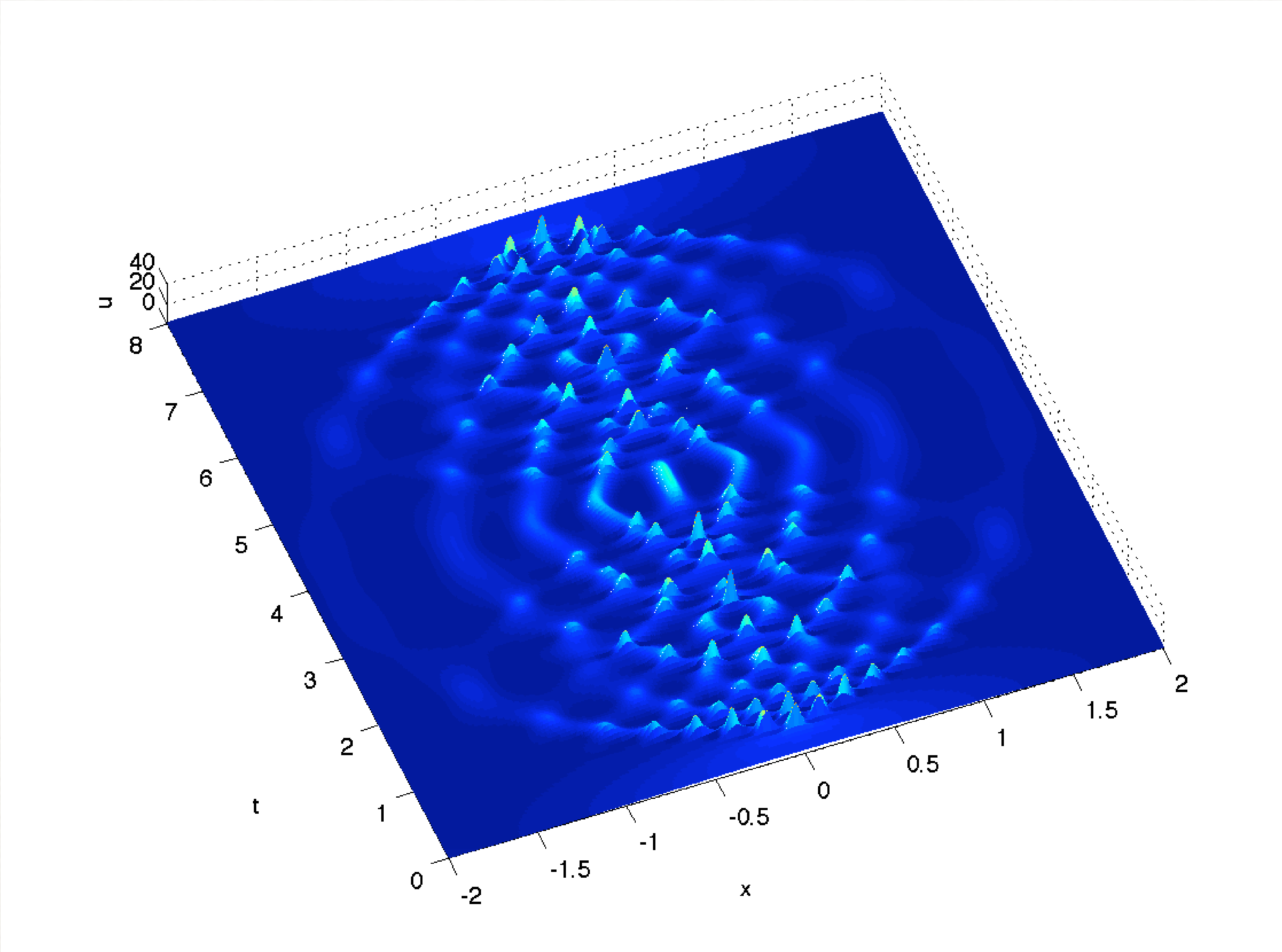
$$\psi_0 = 2\mathrm{sech}x$$



$$u = |\psi|^2$$

- exact solution for $\psi_0 = N\mathrm{sech}x$ in terms of ill conditioned determinant (Satsuma, Yajima 1974),

- rescaling $x \to x/\epsilon$, $t \to t/\epsilon$, here $\psi_0 = 2\mathrm{sech}x$, $\epsilon = 0.2$,



Semiclassical Limit

- NLS equation: focusing ($\rho = -1$) and defocusing ($\rho = 1$)

$$i\epsilon\psi_t + \frac{\epsilon^2}{2}\psi_{xx} - \rho|\psi|^2\psi = 0$$

- slow variables: $u = |\psi|^2, \quad v = \epsilon\Im((\ln\psi)_x)$

$$u_t + (uv)_x = 0$$

$$v_t + vv_x + \rho u_x + \frac{\epsilon^2}{4}\left(\frac{u_x^2}{2u^2} - \frac{u_{xx}}{u}\right)_x = 0$$

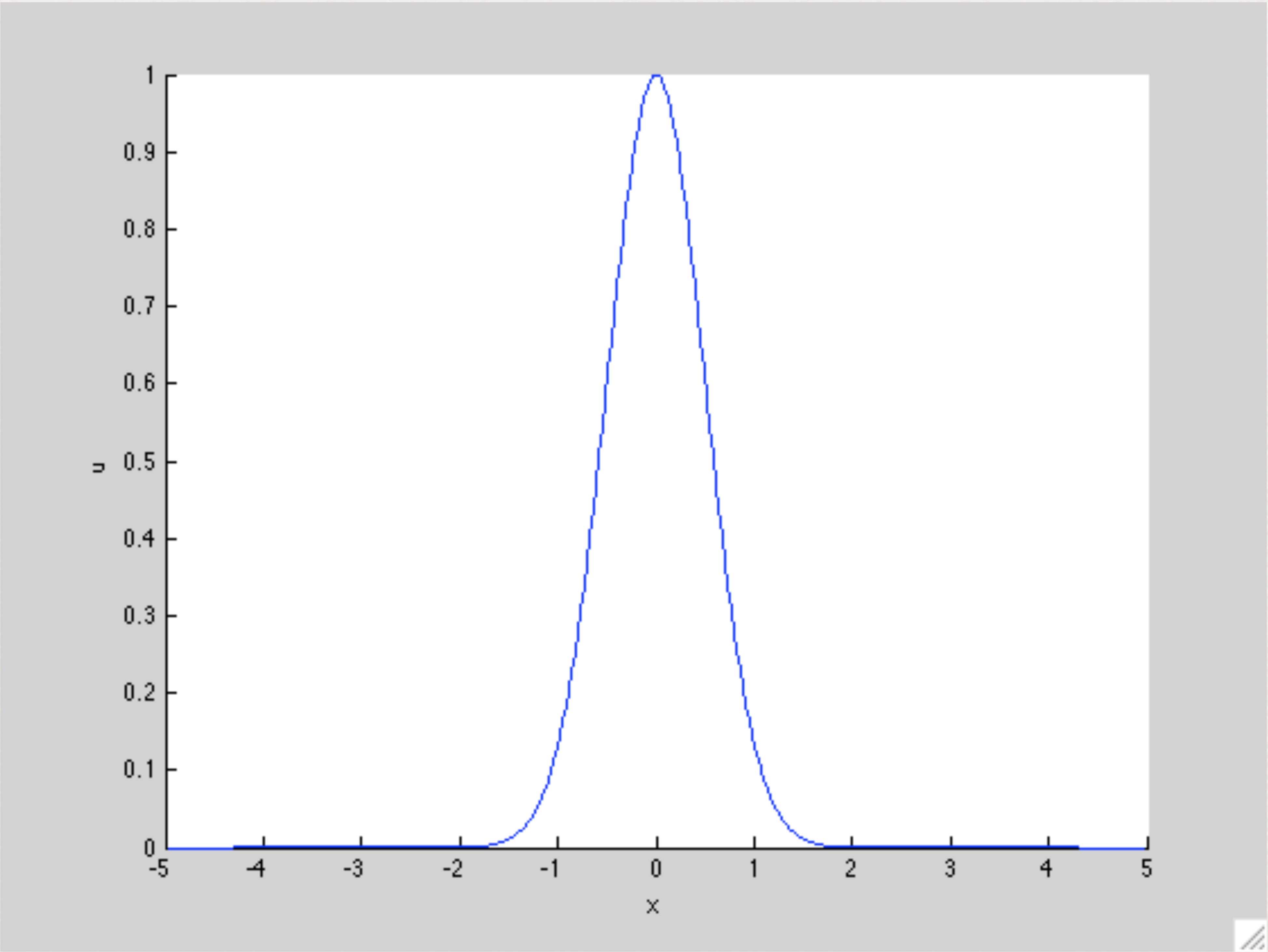
Defocusing NLS

$$\psi_0(x) = \exp(-x^2),$$

$$\epsilon = 0.5,$$

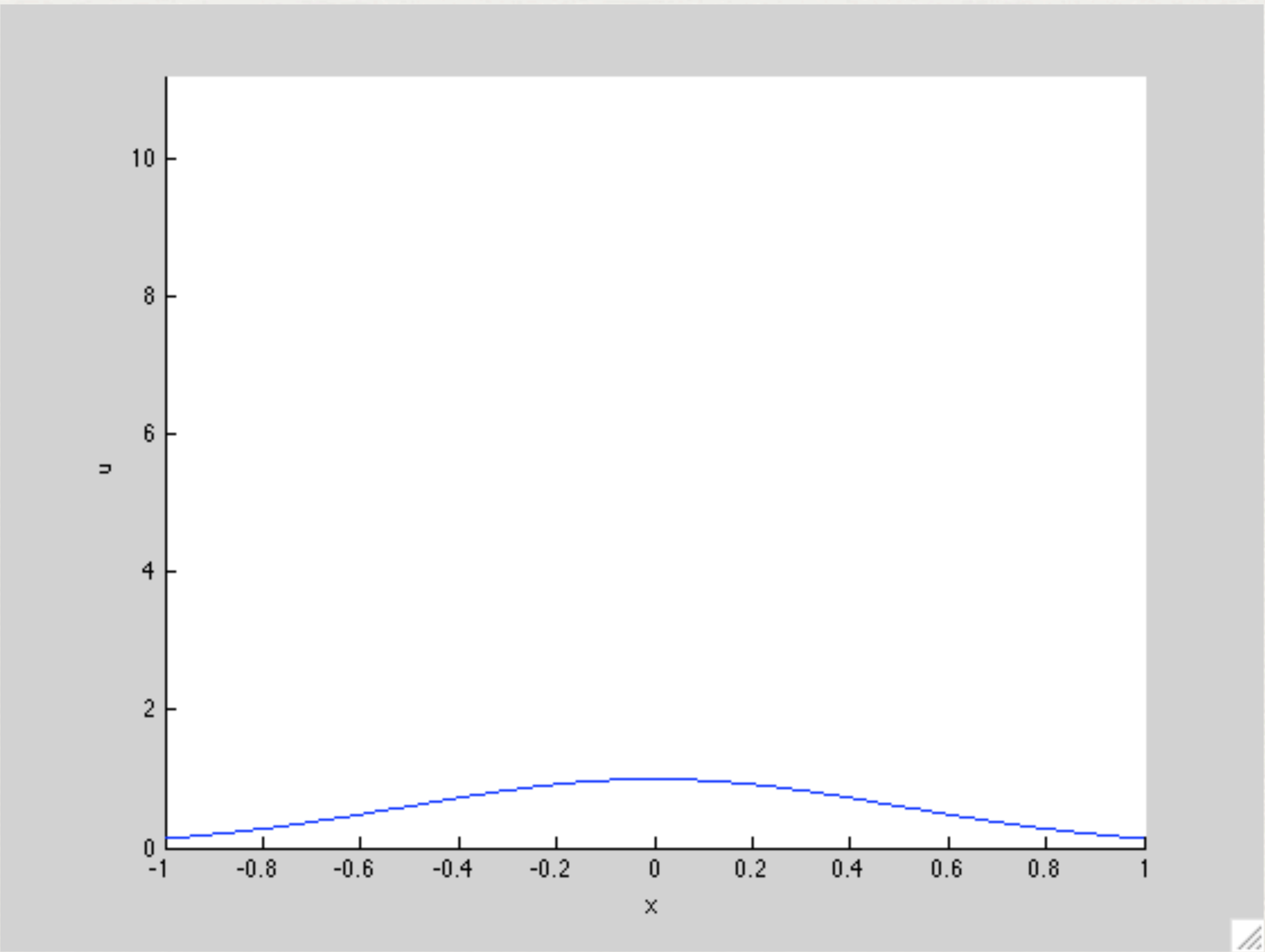
$$0 \leq t \leq 1,$$

$$u = |\psi|^2$$



Focusing NLS

$$\psi_0(x) = \exp(-x^2),$$
$$\epsilon = 0.1,$$
$$0 \leq t \leq 0.8,$$
$$u = |\psi|^2$$



- semiclassical limit $\epsilon \to 0$: hydrodynamic system

$$\begin{aligned} u_t + (uv)_x &= 0 \\ v_t + vv_x + \rho u_x &= 0 \end{aligned}$$

- coefficient matrix: $\begin{pmatrix} v & u \\ \rho & v \end{pmatrix}$, eigenvalues $\lambda = v \pm \sqrt{\rho u}$

- defocusing case similar to KdV, zeroth order and Whitham equations hyperbolic (Jin, Levermore, McLaughlin 1999)

- focusing case: zeroth order and Whitham equations elliptic, only special initial data studied:

  Kamvissis, McLaughlin, Miller 2003: $\psi_0 = A\mathrm{sech}x$

  Tovbis, Venakides, Zhou 2004: $\psi_0 = -\mathrm{sech}x \exp(-i\mu \ln \cosh x)$

Hadamard example

- Cauchy problem for the 2d Laplace equation ($k = 1, 2, 3, \ldots$)

$$\begin{aligned} \Delta_2 u_k &= 0, \quad (x, y) \in \Omega = (0,1) \times (0,1) \\ u_k &= 0, \quad x \in (0,1) \\ u_{k,y} &= \phi_k = (\sin \pi k x)/(\pi k), \quad x \in (0,1) \end{aligned}$$

- unique solution

$$u_k = \frac{\sinh(\pi k y) \sin(\pi k x)}{\pi^2 k^2}$$

- $k \to \infty$: $\phi_k \to 0$, solution u_k trivial. But for small y, strongly oscillating solution with growing amplitude, no convergence.

Zeroth order solution

- Let $f(u, v)$ be a solution to the PDE

$$f_{vv} + u f_{uu} = 0$$

with $f_{uu}^2 + f_{vv}^2 \neq 0$ for u_0, v_0 with $f_v(u_0, v_0) = 0$. Then the solution to

$$x = vt + f_u, \quad 0 = tu + f_v$$

solves the zeroth order system for sufficiently small t.

- point of gradient catastrophe:

$$f_{uu} = f_{vv} = 0, \quad f_{uv} = -t_c$$

Example

- initial data $\psi_0 = A\mathrm{sech}x$

$$f_u = \Re\left[\mathrm{arsinh}\left(\frac{-\frac{1}{2}v + iA}{\sqrt{u}}\right)\right], \quad f_v = -\Re\left[\sqrt{(-\frac{1}{2}v + iA)^2 + u}\right]$$

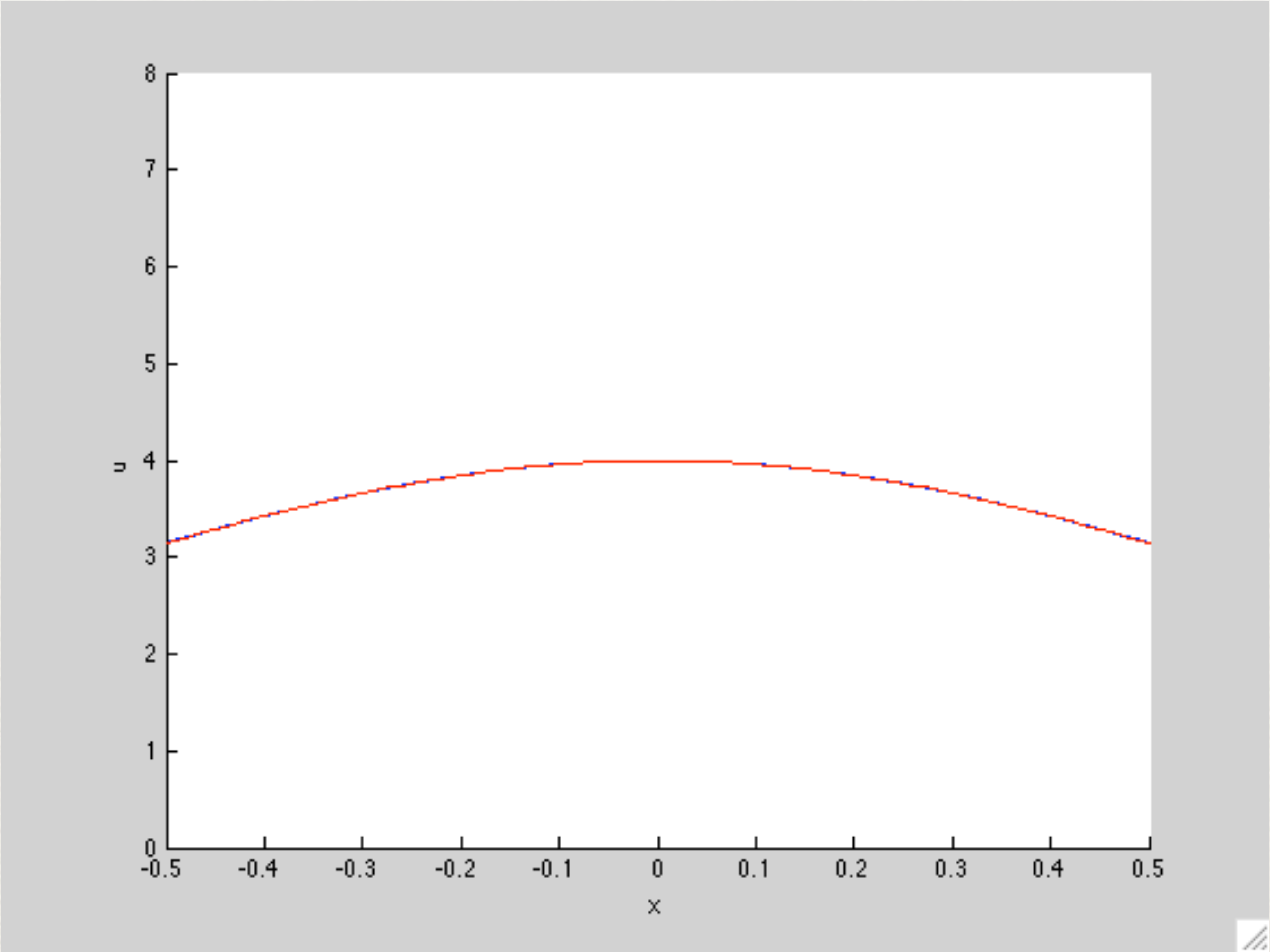
- critical point:

$$u_c = 2A^2, \quad v_c = 0, \quad x_c = 0, \quad t_c = \frac{1}{2A}$$

- cusp formation (α, β constant)

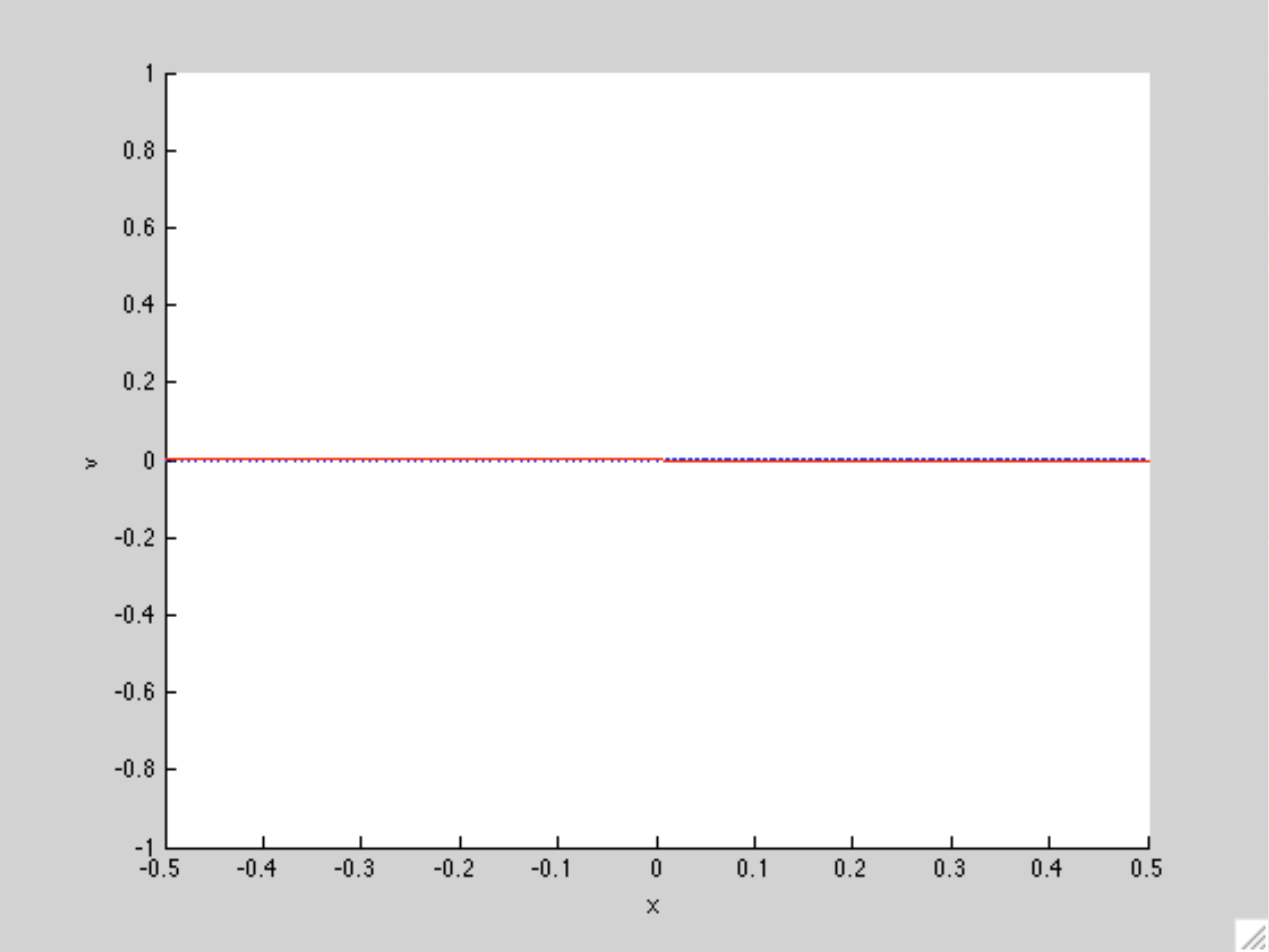
$$u = u_c - \alpha\sqrt{|x|} + \ldots, \quad v = \beta\mathrm{sign}(x)\sqrt{|x|} + \ldots$$

Absolute value



$$\epsilon = 0.04$$

Phase v



$\epsilon = 0.04$

Scaling

- NlS and asymptotic solutions for several values of ϵ ($\epsilon = 0.1, 0.09, \ldots, 0.03, 0.025$),
  linear regression analysis for the L_∞ norm
  of the difference between the solutions
- for $t \ll t_c$: $|u - u^0| = 0(\epsilon^2)$
- for $t = t_c$: largest difference at the critical point
  $|u - u^0| = 0(\epsilon^{2/5})$

Near breakup

- Conjecture: solution close to the critical point given in terms of a solution Ω_0 to the Painlevé I equation

$$\Omega_{\zeta\zeta} = 3\Omega^2 - \zeta$$

in the form

$$u + i\sqrt{u_c}v \approx u_c + i\sqrt{u_c}v_c - \tilde{t}re^{i\phi} + \epsilon^{2/5}(6r\sqrt{u_c})^{2/5}e^{2i\phi/5}\Omega_0(\zeta) + 0(\epsilon^{4/5})$$

where

$$\zeta = \left(\frac{6r}{u_c^2}\right)^{1/5} \frac{1}{\epsilon^{4/5}}\left[-u_c\tilde{t} + i\sqrt{u_c}\tilde{x} + \frac{1}{2}r\tilde{t}^2\right], \quad re^{i\phi} = \frac{1}{f_{uuv}^c + i\sqrt{u_c}f_{uuu}^c}$$

with $\tilde{t} = t - t_c$, $\tilde{x} = x - x_c$.

Painlevé I and *tritronquées* solutions

- Painlevé I:

$$\Omega_{\zeta\zeta} = 3\Omega^2 - \zeta$$

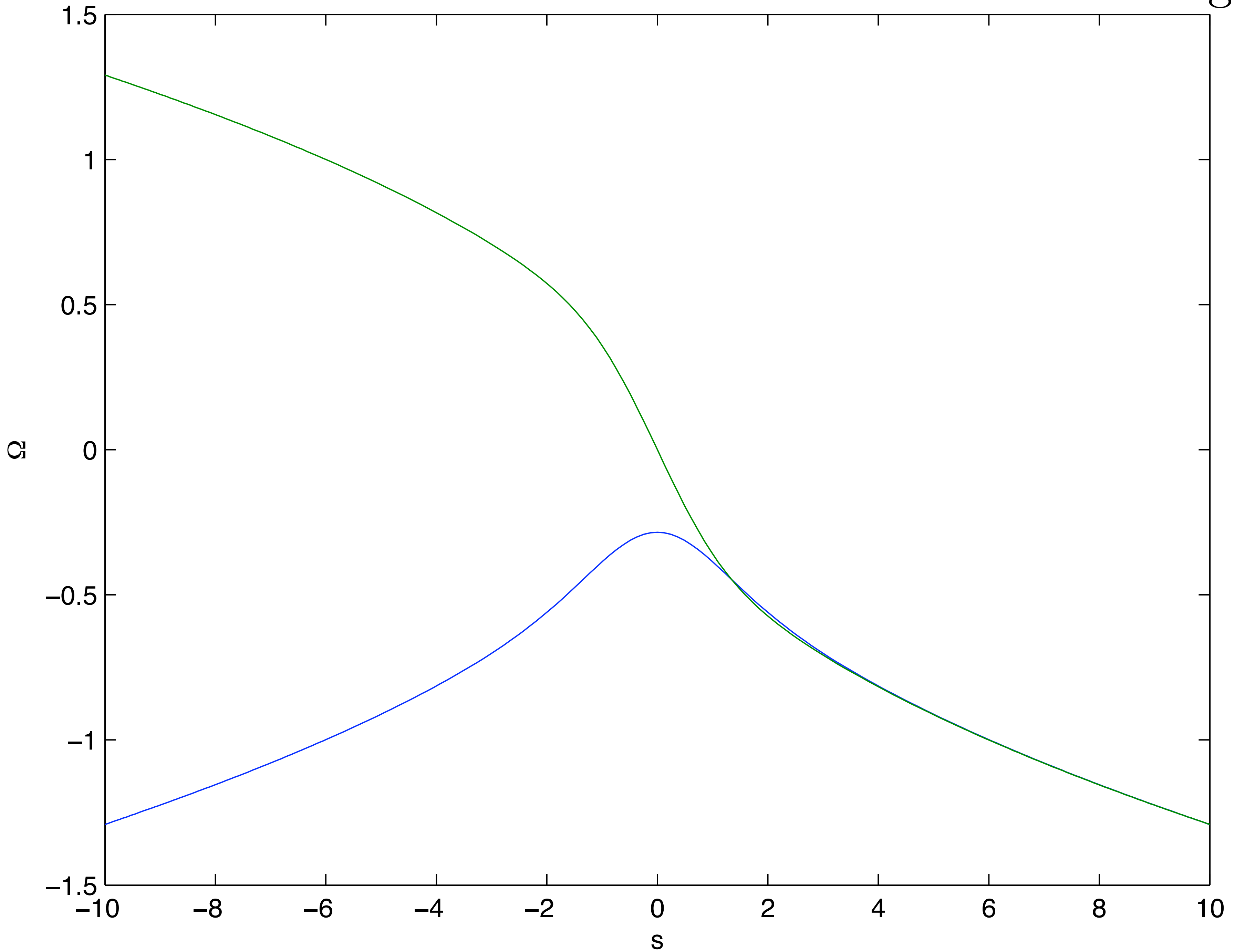
- Boutroux 1913: asymptotic analysis, solutions asymptotically without poles for $|\arg\zeta| < 4\pi/5$.
- Joshi, Kitaev 2001: no pole for $\Re\zeta > -2.38\ldots$
- numerical solution: asymptotic series (divergent)

$$\Omega = \frac{\sigma}{\sqrt{3}}X + \sum_{n=0}^{\infty} \frac{\sigma a_n}{X^{5n-1}}, \quad X = \sqrt{z}, \quad \sigma = \pm 1$$

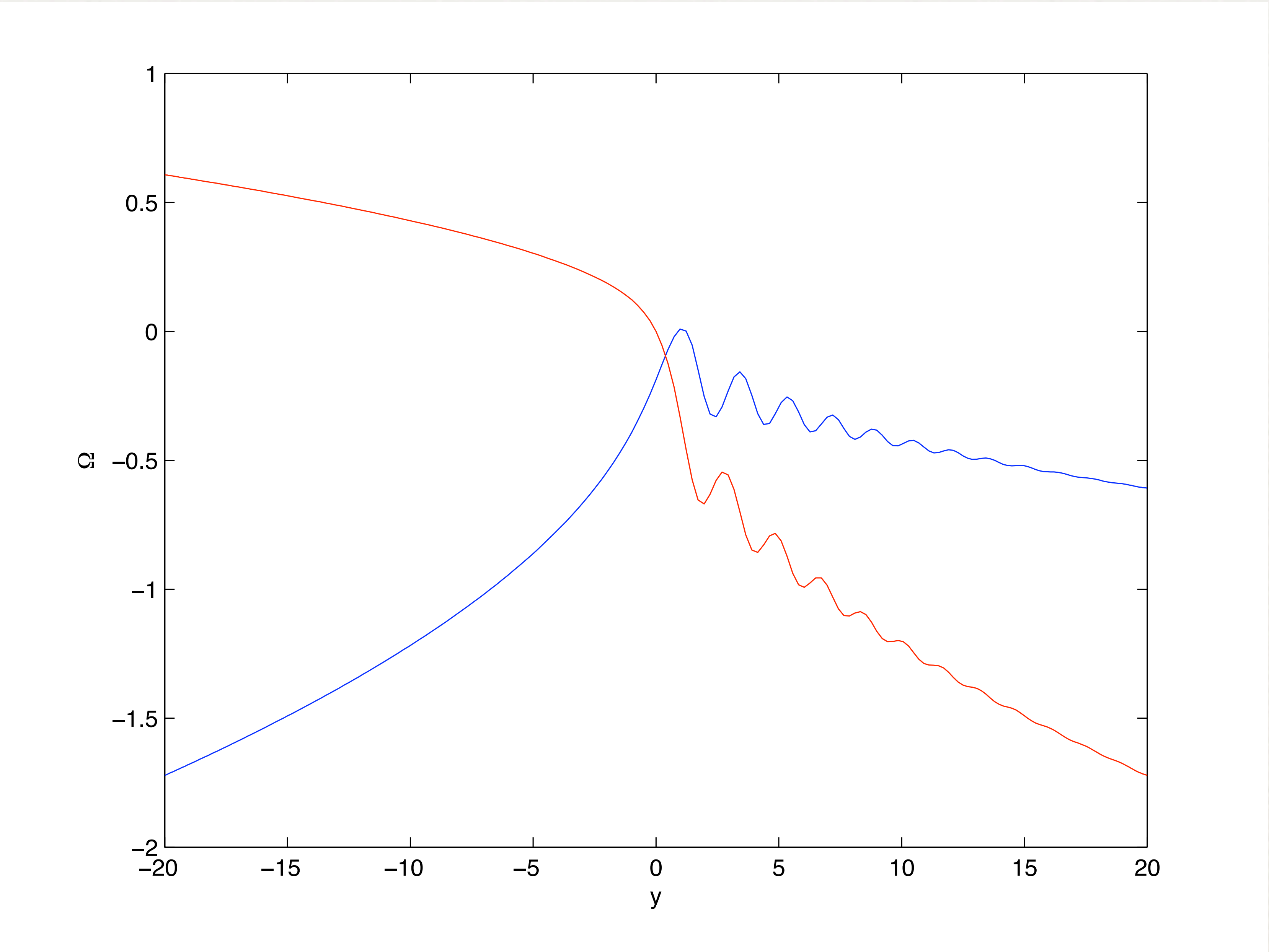
boundary conditions for finite ζ along a straight line, ODE solver *bvp4* in Matlab (collocation method, iteration)

tritronquées solution

solution on the imaginary axis
real part blue,
imaginary part green

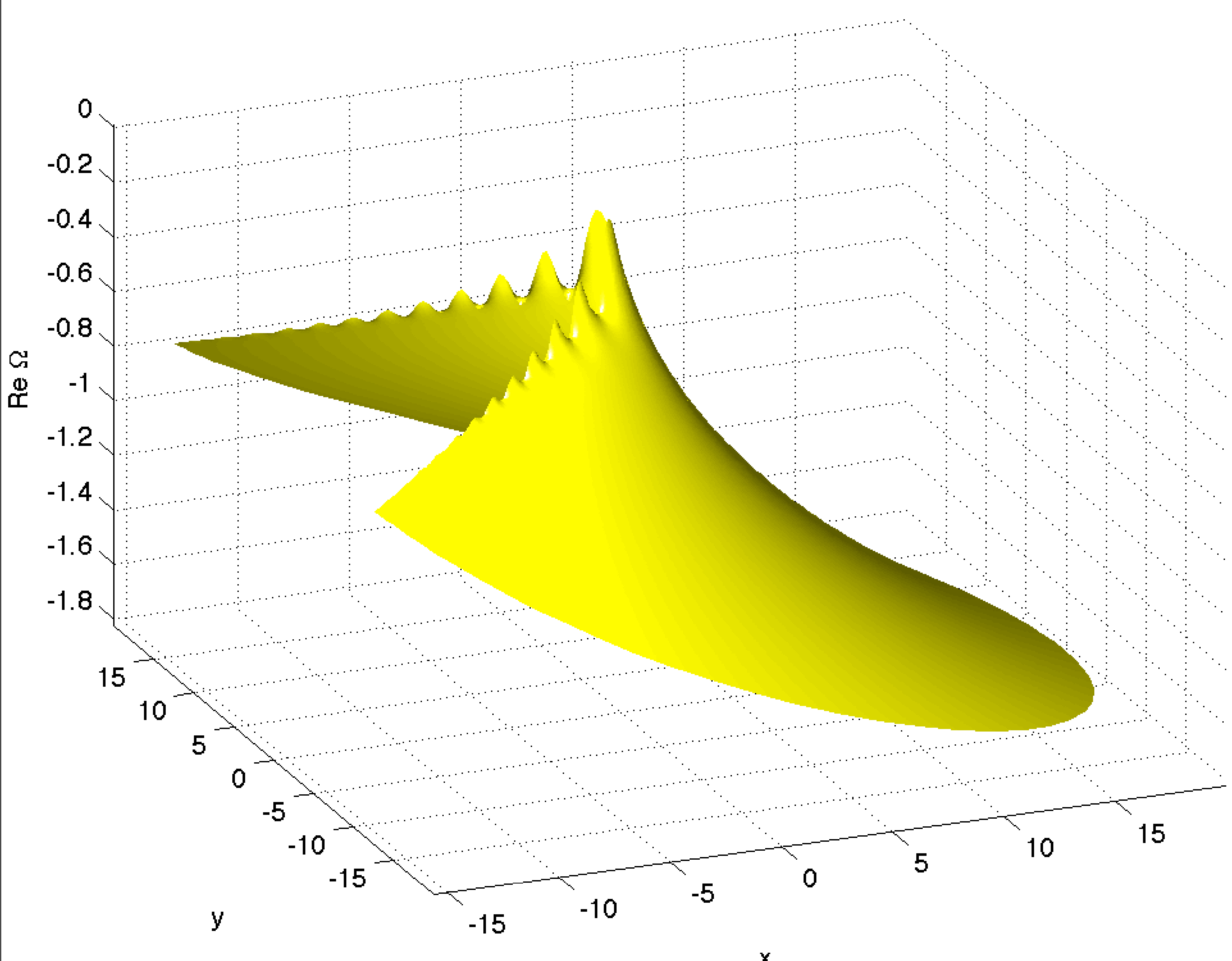


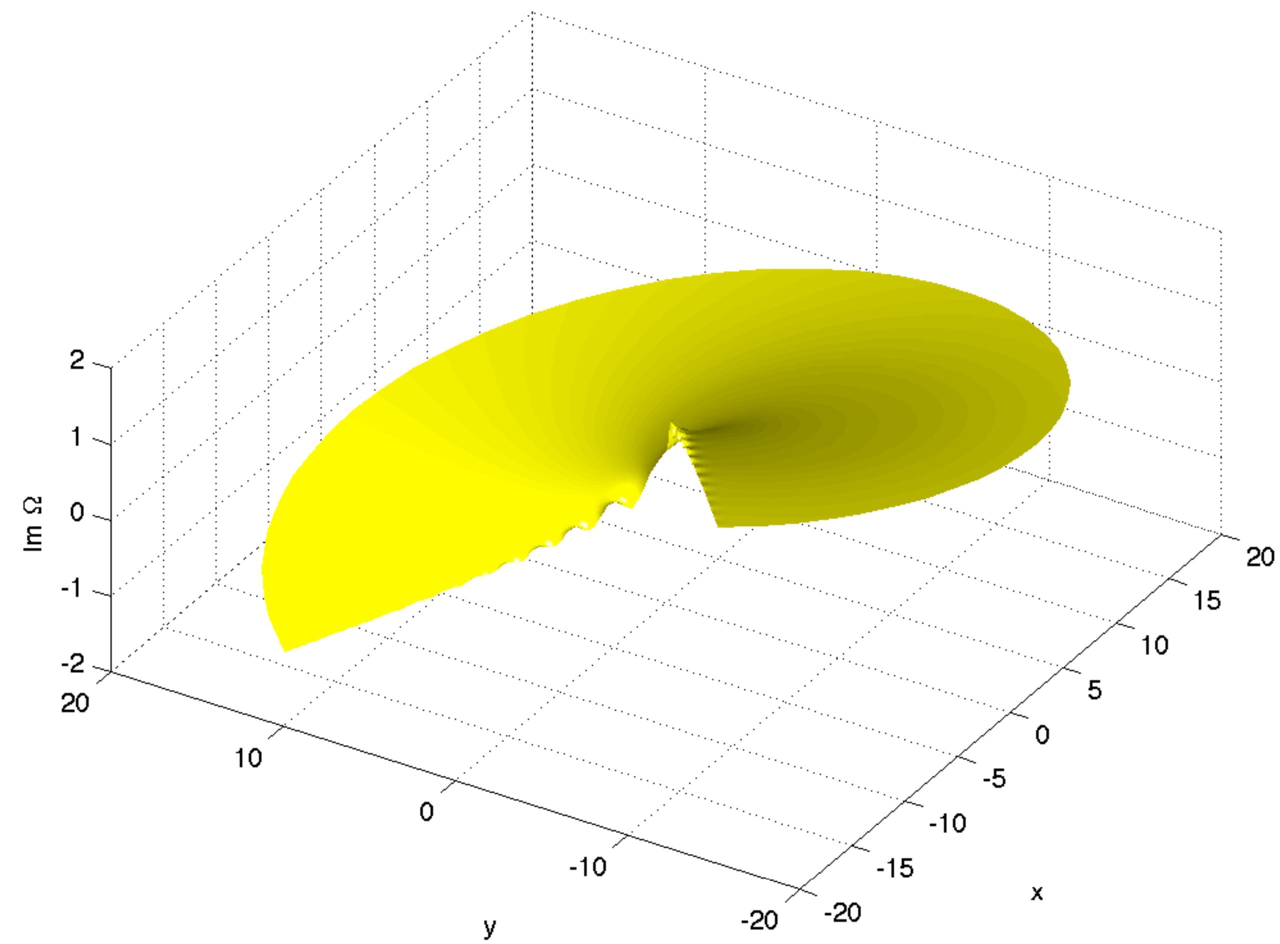
tritronquées solution



- Real (blue) and imaginary part (red) of the *tritronquée* solution close to the critical line (for $\zeta = \exp(i(4\pi/5 - .05))y$) with oscillations of slowly decreasing amplitude

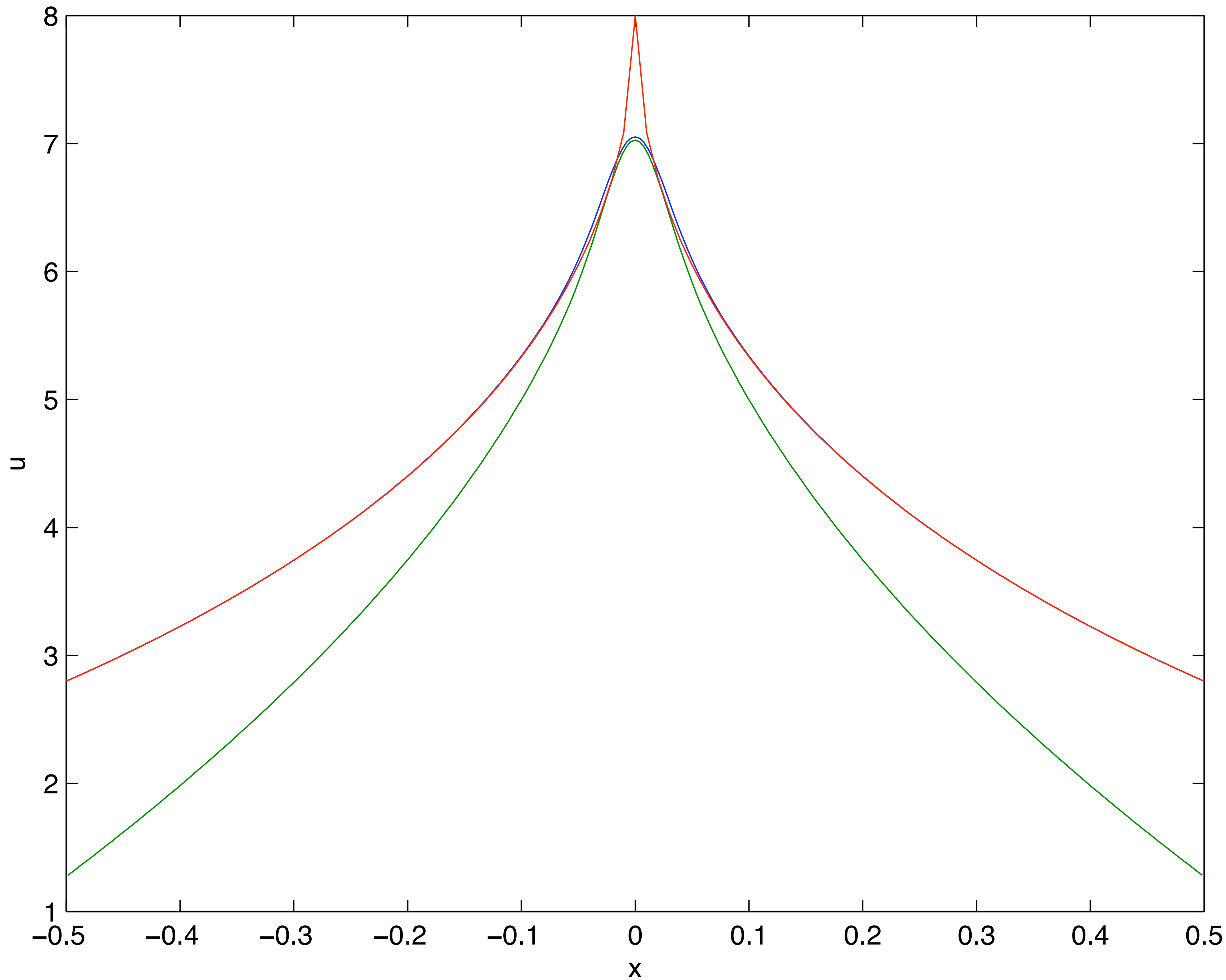
Conjecture: no poles in the sector $arg(z) < 4\pi/5$



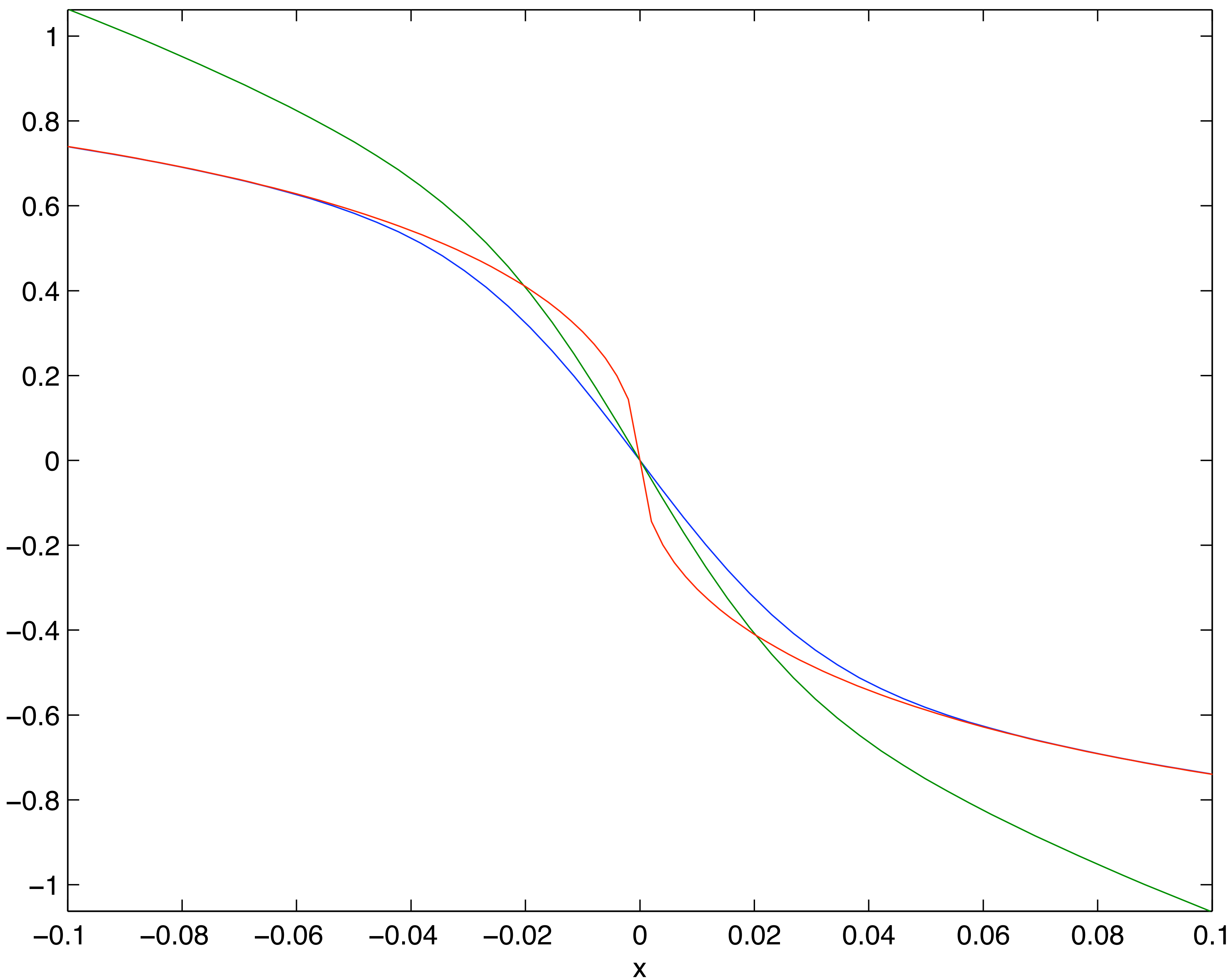


- harmonic function with *tritronquée* boundary data

Critical point

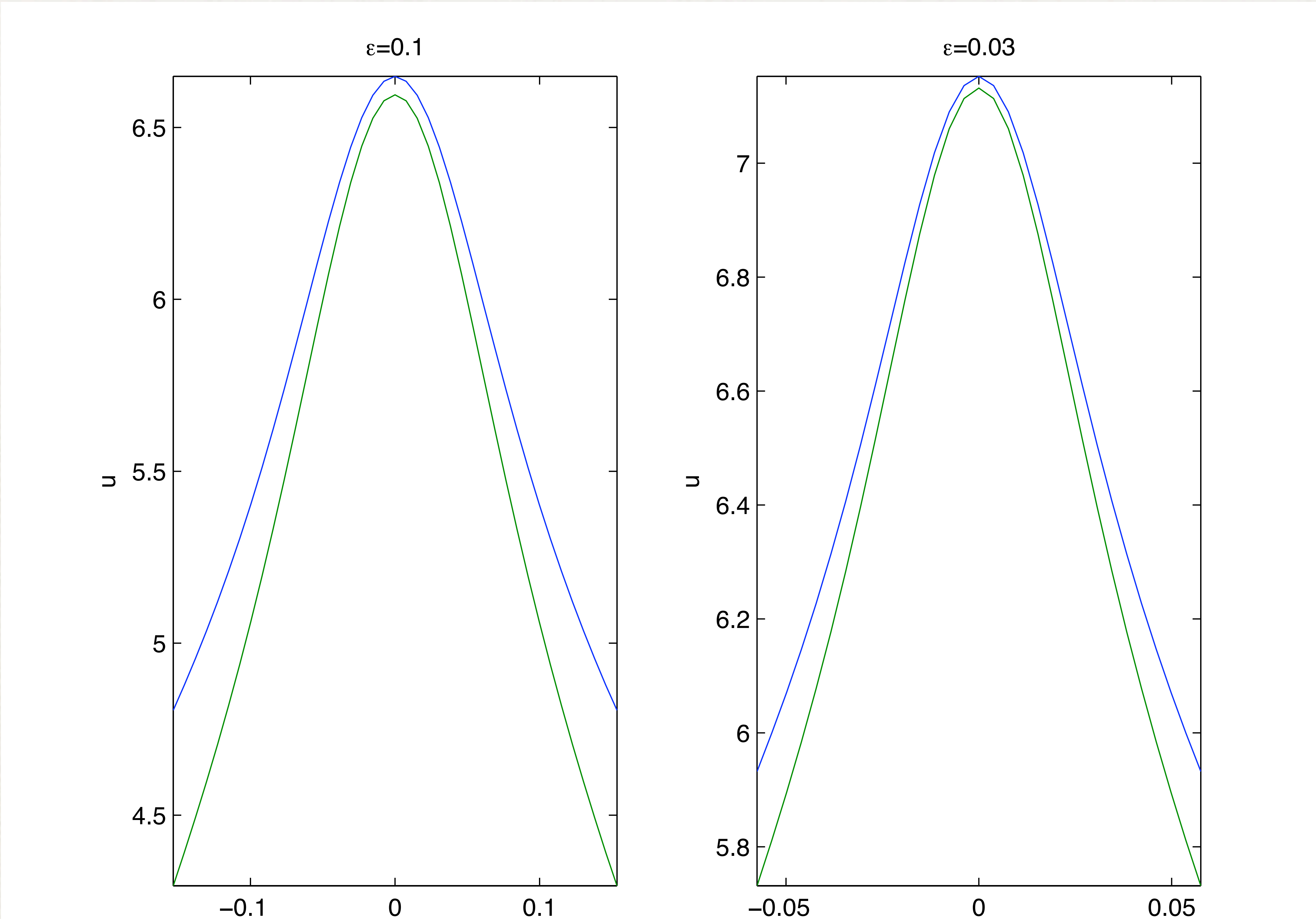


Critical point



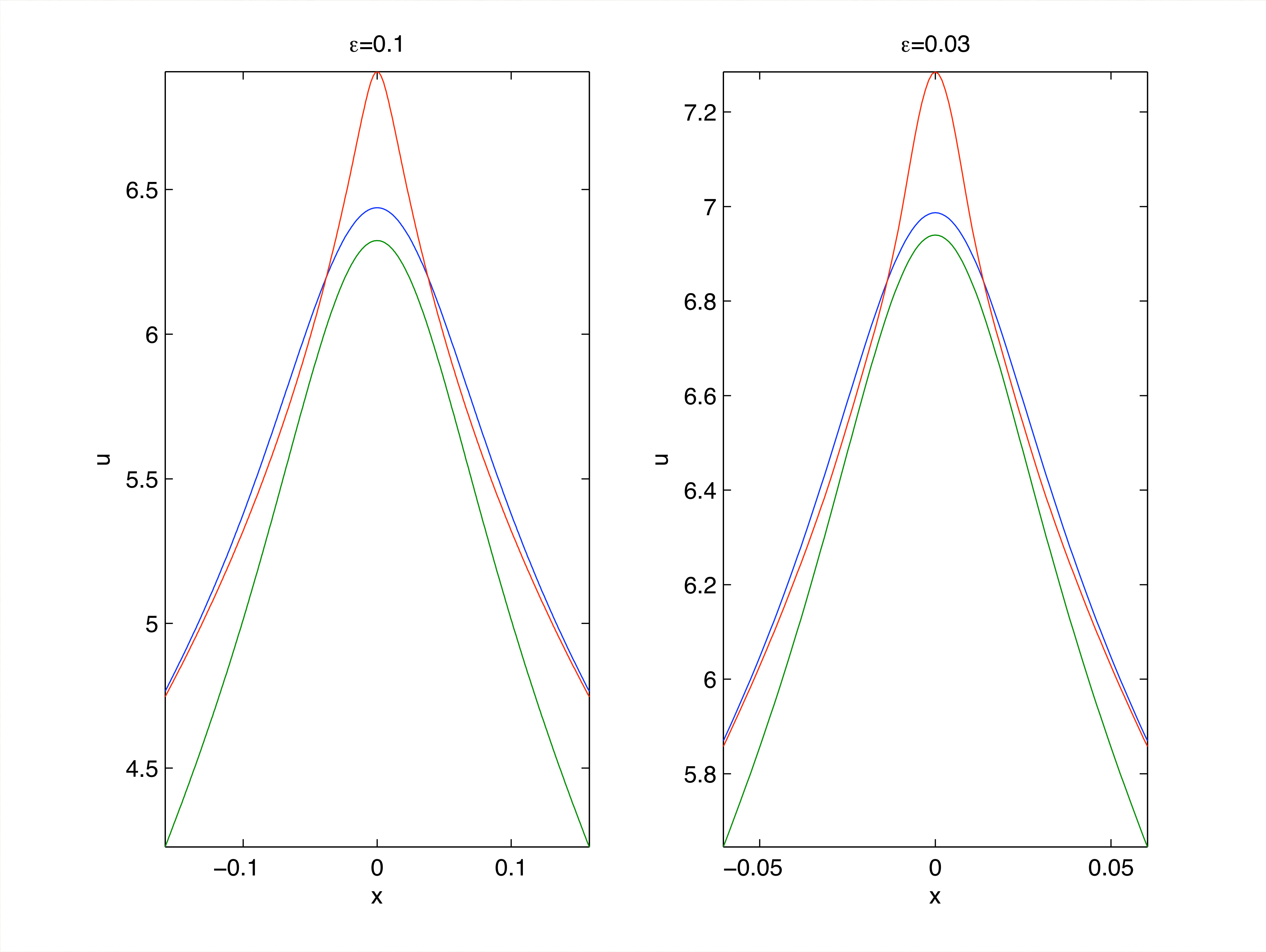
Scaling at breakup

$$|u - u_{tri}| \propto \epsilon^{4/5}$$



Before breakup

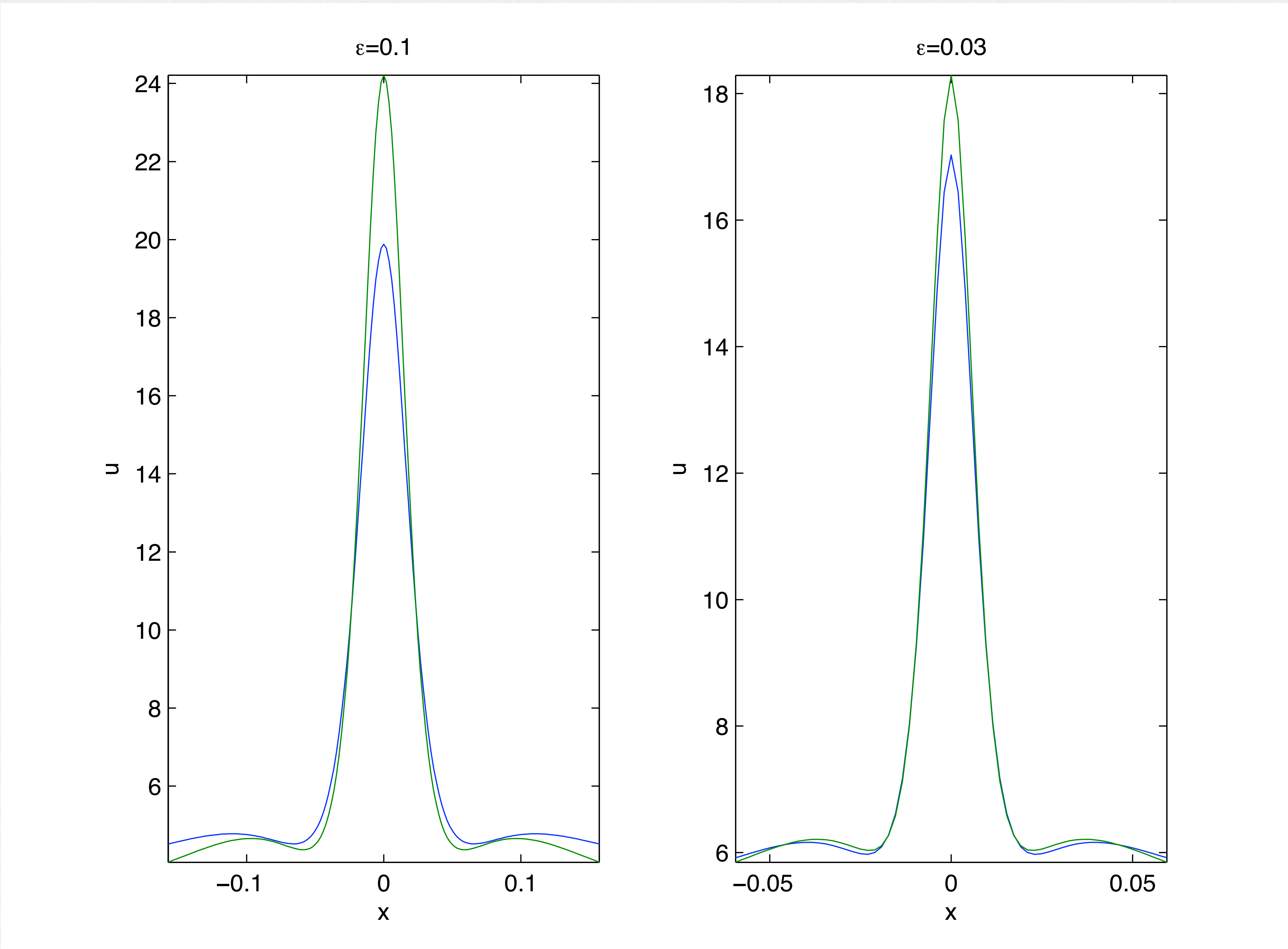
$$t_-(\epsilon) = t_c + u_0/r - \sqrt{(u_0/r)^2 - \epsilon^{4/5}\beta}$$



$\beta = 0.01$

After breakup

$$t_+(\epsilon) = t_c + u_0/r - \sqrt{(u_0/r)^2 + \epsilon^{4/5}\beta}$$



$\beta = 0.1$

Numerical methods

- task: resolve steep gradients in rapid oscillations
- Fourier series for spatial coordinates, method of lines
- fourth-order time stepping to avoid aliasing, integrating factor method (fourth-order Runge-Kutta), exponential time differencing, sliders (Driscoll), time splitting
- Krasny filtering (modulational instability) or more than double precision (>0.025)

- Fourier space: equation of the form

$$U_t = cU + N[U]$$

here: U vector (1+1) or matrix (2+1), c array, $N[U]$ convolution, steep gradients: high frequency terms in c lead to large absolute values despite small ϵ

- exponential time differencing: time discretization and integration with integrating factor

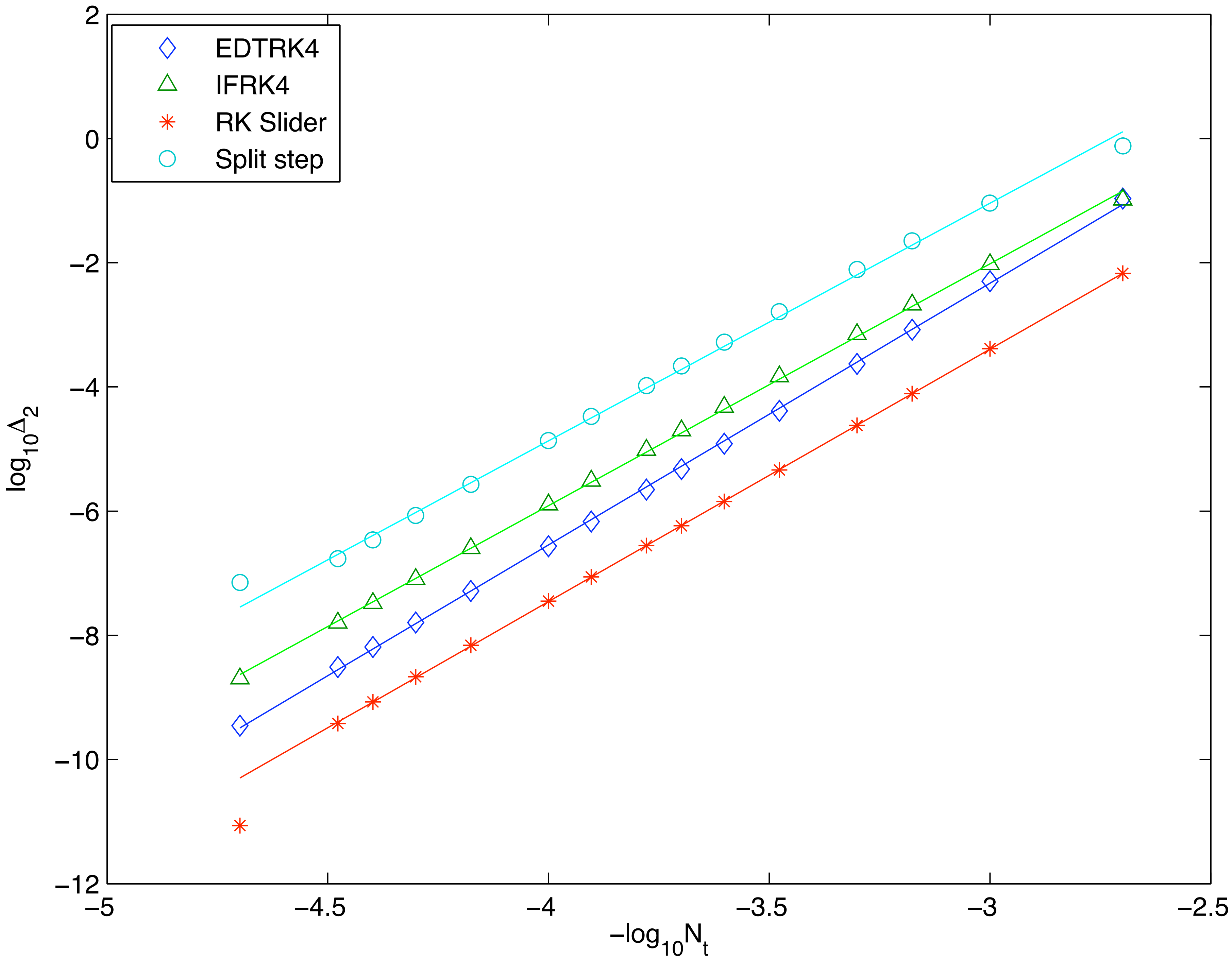
$$U(t_n + h) = e^{ch} U(t_n) + \int_0^h d\tau e^{c(h-\tau)} N[U(t_n + \tau)]$$

fourth-order Runge-Kutta scheme (Cox-Matthews), coefficients via contour integrals (Kassam-Trefethen)

- integrating factor, fourth-order Runge-Kutta (e.g. Trefethen):

$$\left(e^{-ct} U\right)_t = e^{-ct} N[U]$$

Performance

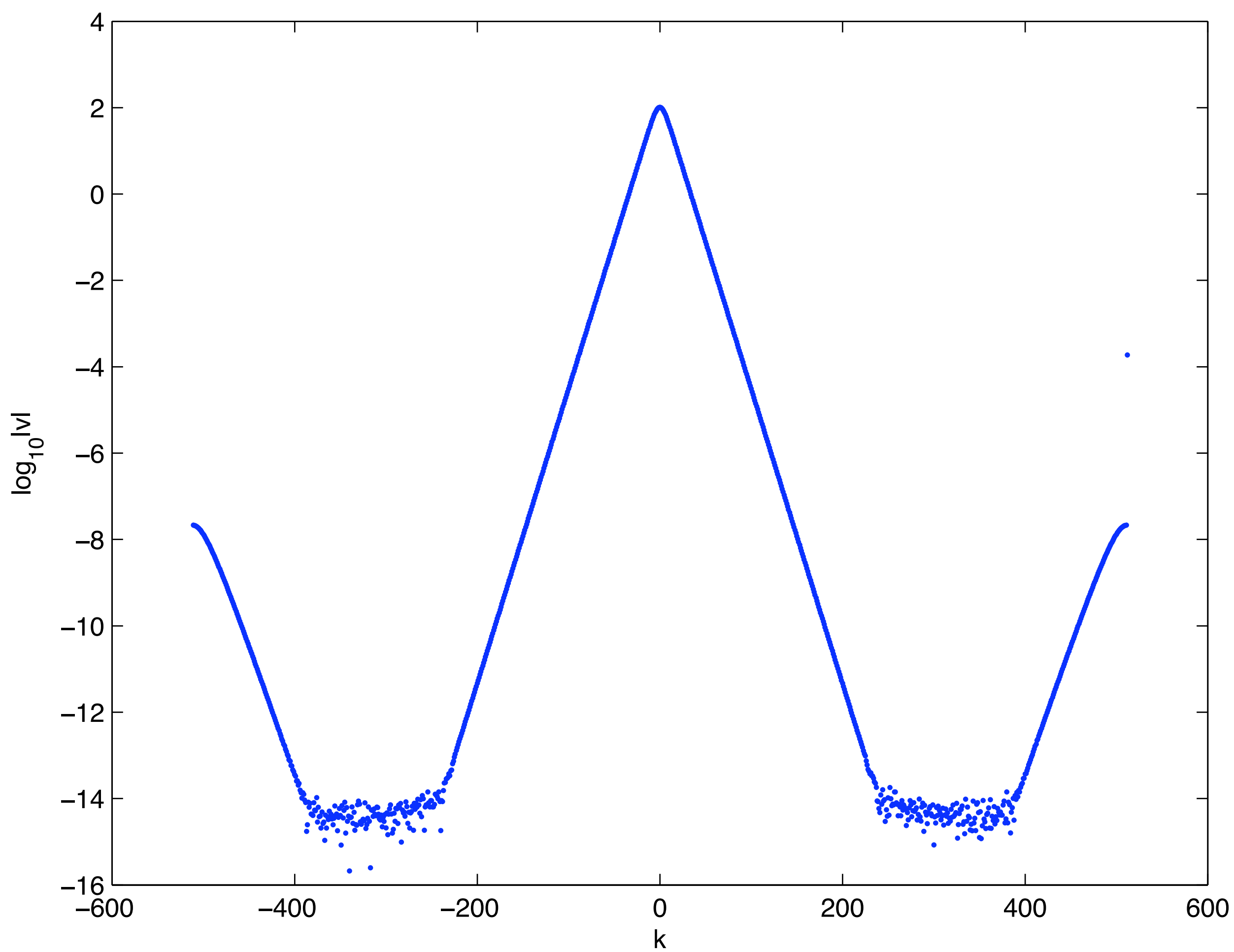


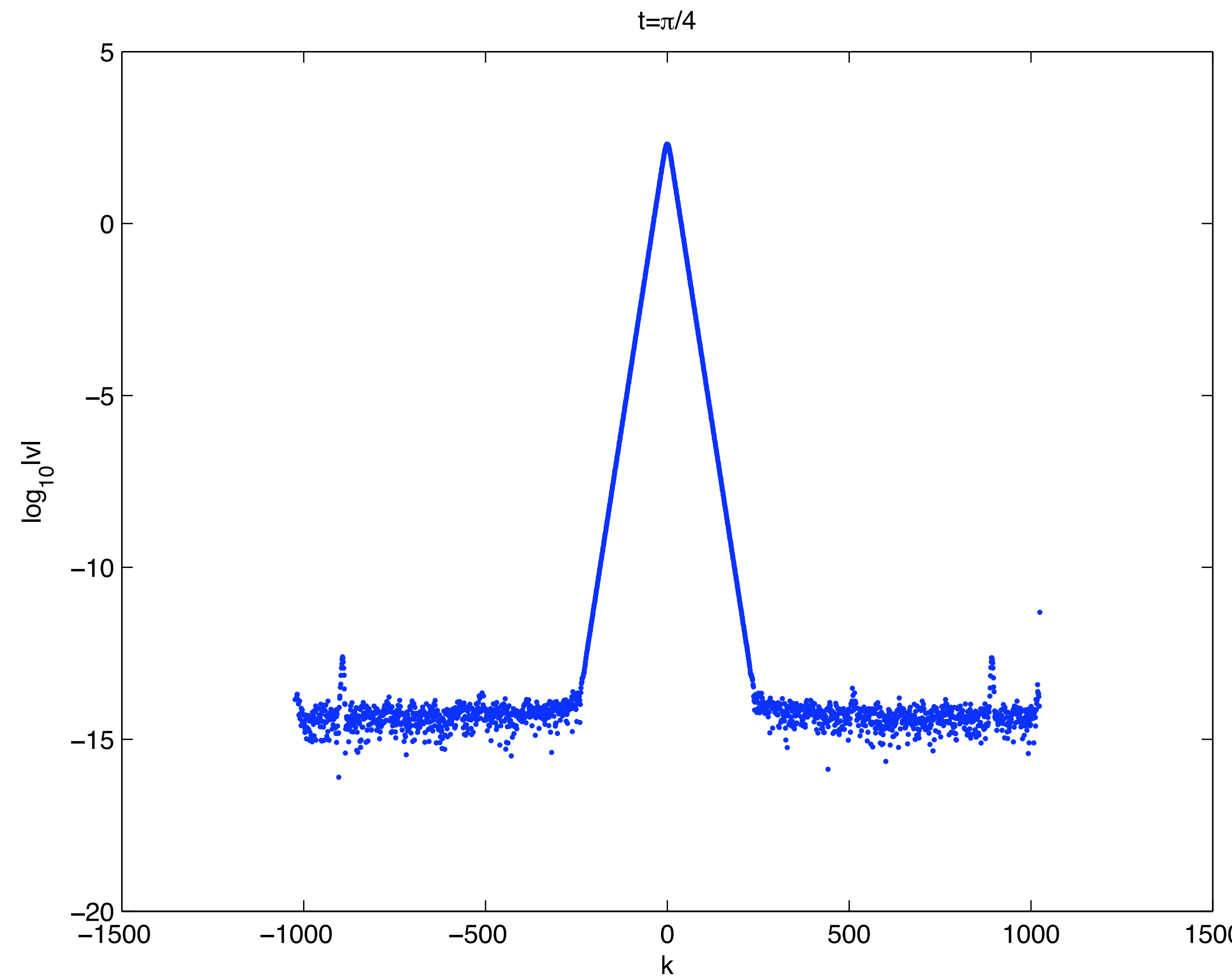
$\psi_0 = \exp(-x^2)$

$\epsilon = 0.1$

$t \leq 0.8$

Modulational instability: breather, t = π/4





breather, t = π/8