Poncelet Porisms and Beyond

Vladimir Dragović

IHES, Paris / Mathematical Institute SANU, Belgrade

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- V. Dragović, M. Radnović, *Hyperelliptic Jacobians as Billiard Algebra of Pencils of Quadrics: Beyond Poncelet Porisms*, Advances in Mathematics **219** (2008) // arXiv:0710.3656
- V. Dragović, M. Radnović, *Geometry of integrable billiards and pencils of quadrics*, Journal de Mathématiques Pures et Appliquées **85** (2006)
- V. Dragović, *Multi-valued hyperelliptic continued fractions of generalized Halphen type*, arXiv: 0809.4931

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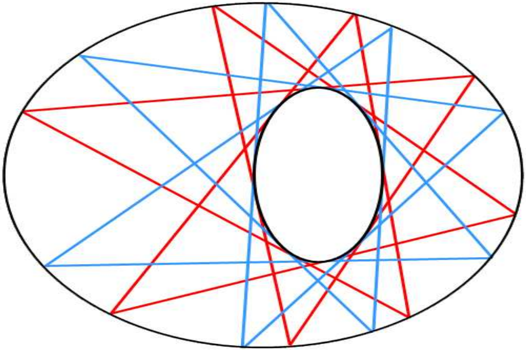
The Poncelet Theorem

Let two conics be given in the plane. If there is a closed polygonal line inscribed in one of them and circumscribed about another one, then there is infinitely many such lines and they all have the same number of edges.



Cayley's Condition

$\mathcal{C} : (Cx, x) = 0$, $\mathcal{D} : (Dx, x) = 0$ – two conics in the projective plane

Cayley's Condition for Even n

There is a polygon with n vertices inscribed in $\mathcal{C}$ and circumscribed about $\mathcal{D}$ if and only if:

$$\begin{vmatrix} C_3 & C_4 & \dots & C_{p+1} \\ C_4 & C_5 & \dots & C_{p+2} \\ & & \dots & \\ C_{p+1} & C_{p+2} & \dots & C_{2p-1} \end{vmatrix} = 0, \quad \text{for} \quad n = 2p,$$

where $\sqrt{\det(C + xD)} = C_0 + C_1 x + C_2 x^2 + \dots$ is the Taylor expansion around $x = 0$.

Cayley's Condition

$\mathcal{C} : (Cx, x) = 0$, $\mathcal{D} : (Dx, x) = 0$ – two conics in the projective plane

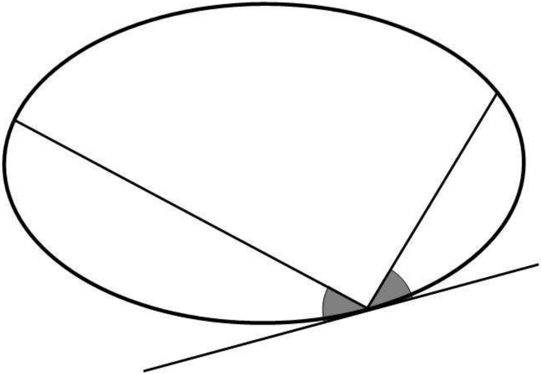
Cayley's Condition for Odd n

There is a polygon with n vertices inscribed in $\mathcal{C}$ and circumscribed about $\mathcal{D}$ if and only if:

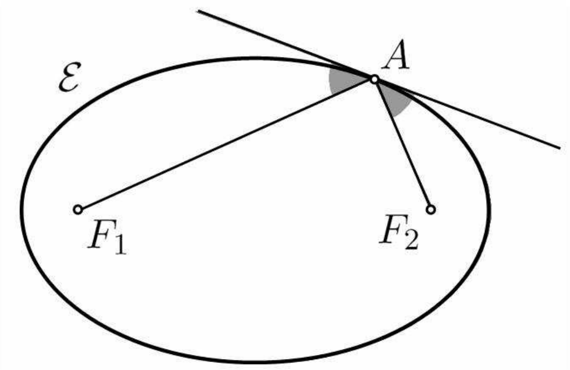
$$\begin{vmatrix} C_2 & C_3 & \dots & C_{p+1} \\ C_3 & C_4 & \dots & C_{p+2} \\ & & \dots & \\ C_{p+1} & C_{p+2} & \dots & C_{2p} \end{vmatrix} = 0 \quad \text{for} \quad n = 2p+1,$$

where $\sqrt{\det(C + xD)} = C_0 + C_1 x + C_2 x^2 + \dots$ is the Taylor expansion around $x = 0$.

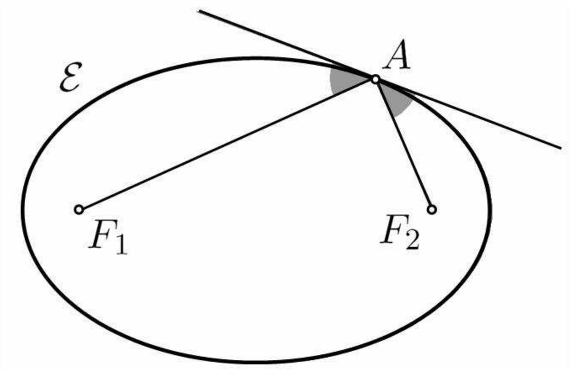
Billiard within Ellipse



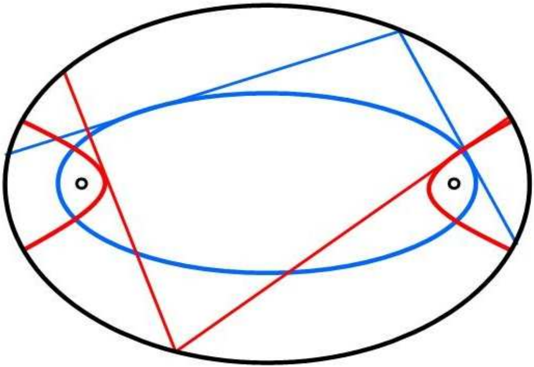
Focal Property of Elliptical Billiard



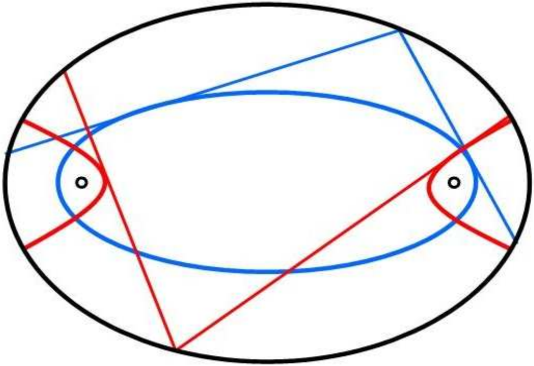
Focal Property of Elliptical Billiard



Caustics of Elliptical Billiard



Caustics of Elliptical Billiard



Periodical Trajectories of Elliptical Billiard

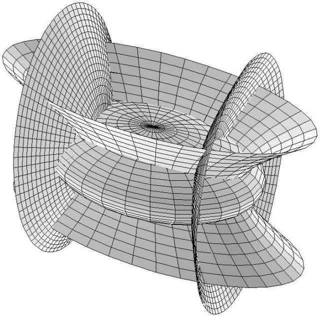
Applied to a pair of confocal conics $\mathcal{C}$, $\mathcal{D}$, the Cayley's condition gives an analytical condition for periodicity of a billiard trajectory within $\mathcal{C}$ with $\mathcal{D}$ as a caustic.

Definition of Confocal Family

A family of confocal quadrics in the d-dimensional Euclidean space $\mathbf{E}^d$ is a family of the form:

$$\mathcal{Q}_\lambda \ : \ \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_d^2}{a_d - \lambda} = 1 \qquad (\lambda \in \mathbf{R}),$$

where $a_1, \ldots, a_d$ are real constants.



Chasles Theorem

Chasles Theorem

Any line in $\mathbf{E}^d$ is tangent to exactly $d-1$ quadrics from a given confocal family. Tangent hyper-planes to these quadrics, constructed at the points of tangency with the line, are orthogonal to each other.

Theorem

Two lines that satisfy the reflection law on a quadric $\mathcal{Q}$ in $\mathbf{E}^d$ are tangent to the same $d-1$ quadrics confocal with $\mathcal{Q}$.

Generalized Poncelet Theorem

Consider a closed billiard trajectory within quadric $\mathcal{Q}$ in $\mathbf{E}^d$. Then all other billiard trajectories within $\mathcal{Q}$, that share the same $d-1$ caustics, are also closed. Moreover, all these closed trajectories have the same number of vertices.

Generalized Cayley Condition

The condition on a billiard trajectory inside ellipsoid $\mathcal{Q}_0$ in $\mathbf{E}^d$, with nondegenerate caustics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_{d-1}}$, to be perodic with period $n \geq d$ is:

$$\operatorname{rank} \begin{pmatrix} B_{n+1} & B_n & \dots & B_{d+1} \\ B_{n+2} & B_{n+1} & \dots & B_{d+2} \\ \dots & & & \\ \dots & & & \\ B_{2n-1} & B_{2n-2} & \dots & B_{n+d-1} \end{pmatrix} < n-d+1,$$

where
$\sqrt{(x-a_1)\dots(x-a_d)(x-\alpha_1)(x-\alpha_{d-1})} = B_0 + B_1 x + B_2 x^2 + \dots$
and all $a_1, \dots, a_d$ are distinct and positive.

Reflection Law in Projective Space

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quadrics that meet transversely. Denote by u the tangent plane to $\mathcal{Q}_1$ at point x and by z the pole of u with respect to $\mathcal{Q}_2$. Suppose lines ℓ_1 and ℓ_2 intersect at x, and the plane containing these two lines meet u along ℓ.

If lines ℓ_1, ℓ_2, xz, ℓ are coplanar and harmonically conjugated, we say that rays ℓ_1 and ℓ_2 **obey the reflection law** at the point x of the quadric $\mathcal{Q}_1$ with respect to the confocal system which contains $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

If we introduce a coordinate system in which quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are confocal in the usual sense, reflection defined in this way is same as the standard one.

One Reflection Theorem

Suppose rays ℓ_1 and ℓ_2 obey the reflection law at x of $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Let ℓ_1 intersects $\mathcal{Q}_2$ at y_1' and y_1, u is a tangent plane to $\mathcal{Q}_1$ at x, and z its pole with respect to $\mathcal{Q}_2$. Then lines $y_1'z$ and y_1z respectively contain intersecting points y_2' and y_2 of ray ℓ_2 with $\mathcal{Q}_2$. Converse is also true.

Corollary

Let rays ℓ_1 and ℓ_2 obey the reflection law of $\mathcal{Q}_1$ with respect to the confocal system determined by quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then ℓ_1 is tangent to $\mathcal{Q}_2$ if and only if is tangent ℓ_2 to $\mathcal{Q}_2$; ℓ_1 intersects $\mathcal{Q}_2$ at two points if and only if ℓ_2 intersects $\mathcal{Q}_2$ at two points.

Next assertion is crucial for proof of the Poncelet theorem.

Double Reflection Theorem

Suppose that $\mathcal{Q}_1$, $\mathcal{Q}_2$ are given quadrics and ℓ_1 line intersecting $\mathcal{Q}_1$ at the point x_1 and $\mathcal{Q}_2$ at y_1. Let u_1, v_1 be tangent planes to $\mathcal{Q}_1$, $\mathcal{Q}_2$ at points x_1, y_1 respectively, and z_1, w_1 their poles with respect to $\mathcal{Q}_2$ and $\mathcal{Q}_1$. Denote by x_2 second intersecting point of the line w_1x_1 with $\mathcal{Q}_1$, by y_2 intersection of y_1z_1 with $\mathcal{Q}_2$ and by ℓ_2, ℓ'_1, ℓ'_2 lines x_1y_2, y_1x_2, x_2y_2. Then pairs ℓ_1, ℓ_2; ℓ_1, ℓ'_1; ℓ_2, ℓ'_2; ℓ'_1, ℓ'_2 obey the reflection law at points x_1 (of $\mathcal{Q}_1$), y_1 (of $\mathcal{Q}_2$), y_2 (of $\mathcal{Q}_2$), x_2 (of $\mathcal{Q}_1$) respectively.

Set $\mathcal{A}_\ell$

$\mathcal{A}_\ell$ – the family of all lines which are tangent to the same $d-1$ quadrics as ℓ

The set $\mathcal{A}_\ell$ is invariant to the billiard reflection on any of the confocal quadrics.

Theorem

For any two given lines x and y from $\mathcal{A}_\ell$, there is a system of at most $d-1$ quadrics from the confocal family, such that the line y is obtained from x by consecutive reflections on these quadrics.

s-skew lines

Definition

For two given lines x and y from $\mathcal{A}_\ell$ we say that they are ***s*-skew** if s is the smallest number such that there exist a system of $s+1 \leq d-1$ quadrics $\mathcal{Q}_k$, $k = 1, \ldots, s+1$ from the confocal family, such that the line y is obtained from x by consecutive reflections on $\mathcal{Q}_k$. If the lines x and y intersect, they are **0-skew**. They are **(-1)-skew** if they coincide.

Weak Poncelet Trajectories

Definition

Suppose that a system S of n quadrics $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ from the confocal family is given. For a system of lines $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_n$ in $\mathcal{A}_\ell$ such that each pair of successive lines $\mathcal{O}_i$, $\mathcal{O}_{i+1}$ satisfies the billiard reflection law at $\mathcal{Q}_{i+1}$ $(0 \le i \le n-1)$, we say that it forms an **s-weak Poncelet trajectory of length n associated to the system S** if the lines $\mathcal{O}_0$ and $\mathcal{O}_n$ are s-skew.

Theorem. The existence of an s-weak Poncelet trajectory of length r is equivalent to:

$$\mathrm{rank}\begin{pmatrix} B_{d+1} & B_{d+2} & \dots & B_{m+1} \\ B_{d+2} & B_{d+3} & \dots & B_{m+2} \\ \dots & \dots & \dots & \dots \\ B_{d+m-s-2} & B_{d+m-s-1} & \dots & B_{r-1} \end{pmatrix} < m-d+1,$$

when $r+s+1=2m$ and

$$\mathrm{rank}\begin{pmatrix} B_{d} & B_{d+1} & \dots & B_{m+1} \\ B_{d+1} & B_{d+2} & \dots & B_{m+2} \\ \dots & \dots & \dots & \dots \\ B_{d+m-s-2} & B_{d+m-s-1} & \dots & B_{r-1} \end{pmatrix} < m-d+2,$$

when $r+s+1=2m+1$.

With $B_0, B_1, B_2, \dots$, we denoted the coefficients in the Taylor expansion of function $y = \sqrt{\mathcal{P}(x)}$ in a neighbourhood of P, where $y^2 = \mathcal{P}(x)$ is the equation of the generalized Cayley curve.

Generalized Weyr's Theorem

Each quadric $\mathcal{Q}$ in $\mathbf{P}^{2d-1}$ contains at most two unirational families of $(d-1)$-dimensional linear subspaces. Such unirational families are usually called rulings of the quadric.

Generalized Weyr's Theorem

Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ be two general quadrics in $\mathbf{P}^{2d-1}$ with the smooth intersection V and $\mathcal{R}_1$, $\mathcal{R}_2$ their rulings. If there exists a closed chain

$$L_1,\ L_2,\ \dots,\ L_{2n},\ L_{2n+1} = L_1$$

of distinct $(d-1)$-dimensional linear subspaces, such that $L_{2i-1} \in \mathcal{R}_1$, $L_{2i} \in \mathcal{R}_2$ $(1 \le i \le n)$ and $L_j \cap L_{j+1} \in F(V)$ $(1 \le j \le 2n)$, then there are such closed chains of subspaces of length $2n$ through any point of $F(V)$.

Generalized Weyr's Chains and Poncelet Polygons

Definition

We will call the chains considered in the Generalized Weyr's theorem generalized Weyr's chains.

Proposition

A generalized Weyr chain of length $2n$ projects into a Poncelet polygon of length $2n$ circumscribing the quadrics $\mathcal{Q}^p_{\alpha_1}, \dots, \mathcal{Q}^p_{\alpha_{d-1}}$ and alternately inscribed into two fixed confocal quadrics (projections of $\mathcal{Q}_1, \mathcal{Q}_2$). Conversely, any such a Poncelet polygon of the length $2n$ circumscribing the quadrics $\mathcal{Q}^p_{\alpha_1}, \dots, \mathcal{Q}^p_{\alpha_{d-1}}$ and alternately inscribed into two fixed confocal quadrics can be lifted to a generalized Weyr chain of length $2n$.

Higher-Dimensional Generalization of the Griffiths-Harris Space Poncelet Theorem

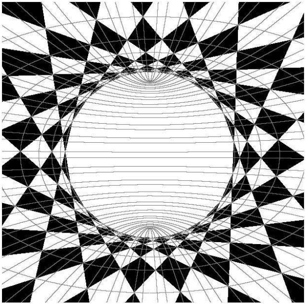
Theorem

Let $\mathcal{Q}_1^*$ and $\mathcal{Q}_2^*$ be the duals of two general quadrics in $\mathbf{P}^{2d-1}$ with the smooth intersection V. Denote by $\mathcal{R}_i, \mathcal{R}_i'$ pairs of unirational families of $(d-1)$-dimensional subspaces of $\mathcal{Q}_i^*$. Suppose there are generalized Weyr's chains between $\mathcal{R}_1$ and $\mathcal{R}_2$ and between $\mathcal{R}_1$ and $\mathcal{R}_2'$. Then there is a finite polyhedron inscribed and subscribed in both quadrics $\mathcal{Q}_1$ and $\mathcal{Q}_2$. There are infinitely many such polyhedra.

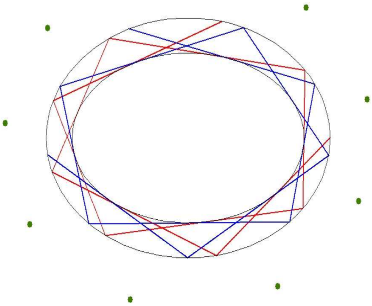
Poncelet-Darboux Grid in Euclidean Plane

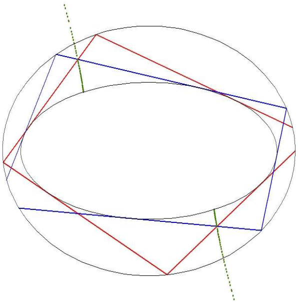
Theorem

Let $\mathcal{E}$ be an ellipse in $\mathbf{E}^2$ and $(a_m)_{m\in\mathbf{Z}}$, $(b_m)_{m\in\mathbf{Z}}$ be two sequences of the segments of billiard trajectories $\mathcal{E}$, sharing the same caustic. Then all the points $a_m \cap b_m$ $(m \in \mathbf{Z})$ belong to one conic $\mathcal{K}$, confocal with $\mathcal{E}$.

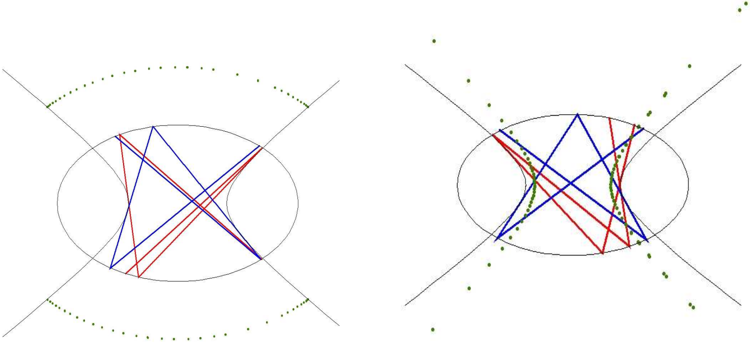


Moreover, under the additional assumption that the caustic is an ellipse, we have: if both trajectories are winding in the same direction about the caustic, then $\mathcal{K}$ is also an ellipse; if the trajectories are winding in opposite directions, then $\mathcal{K}$ is a hyperbola.





For a hyperbola as a caustic, it holds: if segments a_m, b_m intersect the long axis of $\mathcal{E}$ in the same direction, then $\mathcal{K}$ is a hyperbola, otherwise it is an ellipse.



Grids in Arbitrary Dimension

Theorem

Let $(a_m)_{m\in\mathbf{Z}}$, $(b_m)_{m\in\mathbf{Z}}$ be two sequences of the segments of billiard trajectories within the ellipsoid $\mathcal{E}$ in $\mathbf{E}^d$, sharing the same $d-1$ caustics. Suppose the pair (a_0, b_0) is s-skew, and that by the sequence of reflections on quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$ the minimal billiard trajectory connecting a_0 to b_0 is realized.

Then, each pair (a_m, b_m) is s-skew, and the minimal billiard trajectory connecting these two lines is determined by the sequence of reflections on the same quadrics $\mathcal{Q}^1, \dots, \mathcal{Q}^{s+1}$.

Kandinsky, *Grid* 1923.

Continued Fractions

Given a polynomial X of degree $2g+2$ in x. We suppose that X is not a square of a polynomial. Assuming that the values of y and ϵ are finite and fixed, we are going to study HH elements in a neighborhood of ϵ. Then, X can be considered as a polynomial of degree $2g+2$ in s, where $s = x - \epsilon$ is chosen as a variable in a neighborhood of ϵ.

Basic Algebraic Lemma

Let X be a polynomial of degree $2g+2$ in x and $Y = X(y)$ its value at a given fixed point y. Then, there exists a unique triplet of polynomials A, B, C with $\deg A = g+1$, $\deg B = \deg C = g$ in x such that

$$\frac{\sqrt{X}-\sqrt{Y}}{x-y} - C = \frac{B(x-\epsilon)^{g+1}}{\sqrt{X}+A}.$$

Hyperelliptic Halphen-Type Continued Fractions

Factorization of the polynomial B: $B(s) = B_g \prod_{i=1}^{g}(s - t_1^i)$.
Denote $A(t_1^i) = -\sqrt{Y_1^i}$.
Then $\frac{A+\sqrt{X}}{s-t_1^i} = P_A^g(t_1^i, s) + \frac{\sqrt{X}-\sqrt{Y_1^i}}{x-y_1^i}$.
P_A^g is a certain polynomial of degree g in s.
Coefficients of P_A^g depend on the coefficients of A and t_1^i.
Denote $Q_0 = \frac{\sqrt{X}-\sqrt{Y}}{x-y} - C$.
Then we have

$$Q_0 = \frac{B_g \prod_{j=1, j\neq i}^{g}(s - t_1^j)s^{g+1}}{P_A^g(t_1^i, s) + \frac{\sqrt{X}-\sqrt{Y_1^i}}{x-y_1^i}}.$$

Applying Basic Algebraic Lemma we obtain the polynomials $A^{(1,i)}, B^{(1,i)}, C^{(1,i)}$ of degree $g+1, g, g$ respectively, such that

$$\frac{\sqrt{X}-\sqrt{Y_1^i}}{x-y_1^i} - C^{(1,i)} = \frac{B^{(1,i)}(x-\epsilon)^{g+1}}{\sqrt{X}+A^{(1,i)}}.$$

Denote $\alpha_1^{(i)} := P_A^g(t_1^i, s)$, $\beta_1^{(i)} := B_g \prod_{j=1, j\neq i}^{g}(s-t_1^j)s^{g+1}$.

Introduce $Q_1^{(i)}$ by the equation: $Q_0 = \frac{\beta_1^{(i)}}{\alpha_1^{(i)}+Q_1^{(i)}}$.

Observe that $\deg \alpha_1^{(i)} = g$ and $\deg \beta_1^{(i)} = 2g$.

Now, one can go further, step by step:

- factorize $B^{(1,i)}$;
- choose one of its zeroes t_2^j;
- denote by $B^{i,j} := B^{(1,i)}/(s - t_2^j)$.

Denote $\alpha_2^{(i,j)} := P_{A^{1,i}}g(t_2^j, s)$, $\beta_2^{(i,j)} := B^{i,j}s^{g+1}$.

Calculate $Q_2^{(i,j)}$ from the equation

$$Q_1^{(i)} = \frac{\beta_2^{(i,j)}}{\alpha_2^{(i,j)} + Q_2^{(i,j)}}.$$

Following the same scheme, in the i-th step we introduce polynomials:
$A^{(i,j_1,\dots,j_i)}$, $B^{(i,j_1,\dots,j_i)}$, $C^{(i,j_1,\dots,j_i)}$,
with degrees $\deg A = g+1$, $\deg B = g$, $\deg C = g$.

They satisfy the equations:

$$\begin{aligned} A^{(i,j_1,\dots,j_i)} &= C^{(i,j_1,\dots,j_i)}(s - t_i^{j_1,\dots,j_i}) + \sqrt{Y_i^{j_1,\dots,j_i}}, \\ X - A^{(i,j_1,\dots,j_i)2} &= B^{(i,j_1,\dots,j_i)} s^{g+1}(s - t_i^{j_1,\dots,j_i}). \end{aligned}$$

For $g > 1$, the formulae of the $i+1$-th step depend on:

- the choice of one of the roots of the polynomial $B^{(i)}$;
- the choices from the previous steps.

To avoid abuse of notations we omit the indices $j_1, \dots, j_i$, which indicate the choices done in the first i steps, although we assume all the time that the choice has been done.

According to the notations:

$s - t_i | B^{(i-1)}$

$B^{(i-1)} = \frac{\beta_i}{s^{g+1}}(s - t_i)$ or $B^{(i)} = \hat{\beta}_{i+1}(s - t_{i+1})$,

where $\hat{\beta}_i = \beta_i / s^{g+1}$.

We have

$$X - A^{(i-1)2} = \hat{\beta}_i (s - t_{i-1}) s^{g+1} (s - t_i),$$

$$X - A^{(i)2} = \hat{\beta}_{i+1} (s - t_{i+1}) s^{g+1} (s - t_i)$$

$$A^{(i)}(t_i) = \sqrt{Y_i},$$

$$A^{(i-1)}(t_i) = -\sqrt{Y_i}.$$

We introduce λ_i by the relation $A^{(i)}_{g+1} = \sqrt{p_0}\lambda_i$.

Theorem 1

If λ_i is fixed, then t_i and $\{t^{(1)}_{i+1}, \dots, t^{(g)}_{i+1}\}$ are the roots of polynomial equation of degree $g+1$ in s

$$Q_X(\lambda_i, s) = 0.$$

Theorem 2

If t_i is fixed, then λ_i and λ_{i-1} are the roots of the polynomial equation of degree 2 in λ:

$$Q_X(\lambda_{i-1}, t_i) = 0, \qquad Q_X(\lambda_i, t_i) = 0.$$

Periodicity and Symmetry

According to Theorem 2, in the case $t_h = t_k$ for some h, k, there are two possibilities:
(I) $\lambda_{h-1} = \lambda_{k-1}, \qquad \lambda_h = \lambda_k;$
(II) $\lambda_{h-1} = \lambda_k, \qquad \lambda_h = \lambda_{k-1}.$
The first possibility leads to periodicity:

$$t_{h+s} = t_{k+s}, \quad \lambda_{h+s} = \lambda_{k+s}$$

for any s and with appropriate choice of roots.
If $p = h - k$ and $r \cong s(\mod p)$ then $\alpha_r = \alpha_s, \quad \beta_r = \beta_s$.
The second possibility leads to symmetry:

$$t_{h+s} = t_{k-s}, \quad \lambda_{h+s} = \lambda_{k-s-1}$$

for any s.

Definition

(i) If $h+k=2n$ we say that HH c. f. is even symmetric with

$$\alpha_{n-i}=\alpha_{n+i}, \qquad \beta_{n-i}=\beta_{n+i-1}.$$

for any i and with α_n as the centre of symmetry.

(ii) If $h+k=2n+1$ we say that HH c. f. is odd symmetric with

$$\alpha_{n-i}=\alpha_{n+i-1}, \qquad \beta_{n-i}=\beta_{n+i}.$$

for any i and with β_n as the centre of symmetry.

Proposition 1

(A) If a HH c. f. is periodic with the period of $2r$ and even symmetric with α_n as the centre, then it is also even symmetric with respect α_{n+r}.

(B) If a HH c. f. is periodic with the period of $2r$ and odd symmetric with respect β_n, then it is also odd symmetric with respect β_{n+r}.

(C) If a HH c. f. is periodic with the period of $2r-1$ and even symmetric with respect α_n, then it is also odd symmetric with respect β_{n+r}. The converse is also true.

Proposition 2

If a HH c. f. is double symmetric, then it is periodic. Moreover:

(A) If a HH c. f. is even symmetric with respect α_m and α_n, $n < m$ then the period is $2(n - m)$.

(B) If a HH c. f. is odd symmetric with respect β_m and β_n, $n < m$ then the period is $2(m - n)$.

(C) If a HH c. f. is even symmetric with respect α_n and β_m, then the period is $2(n - m) + 1$ in the case $m \leq n$ and the period is $2(m - n) - 1$ when $m > n$.

Observation

(i) A HH c. f. can be at the same time even symmetric and odd symmetric.

(ii) If $\lambda_i = \lambda_{i-1}$ then the symmetry is even; if $t_i = t_{i+1}$ then the symmetry is odd.

Proposition 3

An H. H. c. f. is even-symmetric with the central parameter y if $X(y) = 0$.

Invariant Approach

We pass to the general case, with polynomial X of degree $2g+2$.

Relation $Q_X(\lambda, s) = 0$ defines a basic curve Γ_X.
G – genus of Γ_X

R_e – the ramification points of the projection of Γ_X to the s-plane
We call them even-symmetric points of the basic curve.

R_{o+r} – the ramification points of the projection of Γ_X to the λ-plane
R_{o+r} is are the union of the odd-symmetric points and the gluing points.

From Proposition 3, we get $\deg R_e = 2g + 2$.

By applying the Riemann-Hurvitz formula, we have:
$2 - 2G = 4 - \deg R_e$, $2 - 2g = 2(g+1) - \deg R_{o+r}$.
Thus $genus(\Gamma_X) = G = g$ and $\deg R_{o+r} = 4g$.

We get a birational morphism $f : \Gamma \to \Gamma_X$ by the formulae
$f : (x, s) \mapsto (t, \lambda)$, where
$t = x$, $\lambda = \frac{1}{t^{g+1}} \left(\frac{s}{\sqrt{p_0}} - Q_g(t) \right)$, $Q_g(t) = 1 + q_1 t + \cdots + q_g t^g$.

f satisfies commuting relation $f \circ \tau_\Gamma = \tau_{\Gamma_X} \circ f$, where τ_Γ and τ_{Γ_X} are natural involutions on the hyperelliptic curves Γ and Γ_X respectively.

Multi-valued divisor dynamics

The inverse image of a value z of the function λ is a divisor of degree $g+1$:

$$\lambda^{-1}(z) =: D(z), \qquad \deg D(z) = g+1.$$

The HH-continued fractions development can be described as a multi-valued discrete dynamics of divisors $D_k^j = D(z_k^j)$.
Lower index k denotes the k-th step of the dynamics;
upper index j goes in the range from 1 to $(g+1)k$ denoting branches of multivaluedness.

$$D_0 := D(\lambda(P_0)) = P_0^1 + P_0^2 + \cdots + P_0^{g+1}, \text{ with } \lambda(P_0^i) = \lambda(P_0^j)$$
$$D_1^j := D\left(\lambda(\tau_\Gamma(P_0^j))\right)$$
$$D_{k-1}^j := P_{k-1}^{(j,1)} + \cdots + P_{k-1}^{(j,(g+1))}$$

We get $g+1$ new divisors $D_k^{(j-1)(g+1)+l} := D\left(\lambda(\tau_\Gamma(P_{k-1}^{(j,l)}))\right)$,
$l = 1, \ldots, g+1$

In the case of genus one, this dynamics can be traced out from the $2-2$ - correspondence $Q_\Gamma(\lambda, t) = 0$.

There exist constants a, b, c, d, T such that for every i we have $\lambda_i = \frac{ax(u_i+T)+b}{cx(u_i+T)+d}$, where u is an uniformizing parameter on the elliptic curve.

$u_{i+1} = u_i + 2T$

$\lambda_{i+1} = \frac{ax(u_i+3T)+b}{cx(u_i+3T)+d}$

In the cases of higher genera the dynamics is much more complicated. Thus, we have to pass to the consideration of generalized Jacobians.

Remainders, Continuants and Approximation

We consider an HH c. f. of an element f:

$$f = C + \frac{\beta_1|}{|\alpha_1} + \frac{\beta_2|}{|\alpha_2} + \dots$$

Together with the remainder of rank i Q_i, where $Q_i = \frac{B^{(i)}s^{g+1}}{\sqrt{X}+A^{(i)}}$, we consider:

the continuants (G_i), (H_i) and

the convergents G_i/H_i such that: $\begin{bmatrix} G_m & G_{m-1} \\ H_m & H_{m-1} \end{bmatrix} = T_C T_1 \cdots T_m$.

Here $T_i = \begin{bmatrix} \alpha_i & 1 \\ \beta_i & 0 \end{bmatrix}$ and $T_C = \begin{bmatrix} C & 1 \\ 1 & 0 \end{bmatrix}$.

By taking the determinant of the above matrix relation, we get:

$$G_m H_{m-1} - G_{m-1} H_m = (-1)^{m-1} \beta_1 \beta_2 \dots \beta_m = \delta_m s^{(g+1)m}$$

$$\deg \delta_m = (g-1)m.$$

Proposition 4

The degree of the continuants is $\deg G_m = g(m+1)$, $\deg H_m = gm$.

Theorem 3

The polynomial $G_m H_{m-1} - H_m G_{m-1}$ is of degree $2gm$. The first $(g+1)m$ coefficients are zero.

Theorem 4

If $X(\epsilon) \neq 0$ and $\epsilon \neq y$, then the element $\hat{G}_m - \hat{H}_m\sqrt{X} = G_m - H_m\frac{\sqrt{X}-\sqrt{Y}}{x-y}$ has a zero of order $(g+1)(m+1)$ at $s = 0$.
If $H(0) \neq 0$ then the differences

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} - \frac{G_m}{H_m}, \qquad \sqrt{X} - \frac{\hat{G}_m}{\hat{H}_m}$$

have developments starting with the order of $s^{(g+1)(m+1)}$.