Teichmüller spaces of orbifold Riemann surfaces

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- Poincaré geometry: $\Sigma_{g,s,n} = \mathbb{H}^2_+/\Gamma_{g,s,n}$, the group $\Gamma_{g,s,n}$ a subgroup of $PSL(2,\mathbb{R})$ **almost** Fuchsian;
  - g the genus ($g \geq 0$)
  - s the number of holes ($s \geq 1$)
  - n the number of $\mathbb{Z}_2$-orbifold points ($n \geq 0$)
- observables = lengths of closed geodesics
- triangulation and Teichmüller space coordinates a'la Penner and Fock (technique of fat graphs)
- Poisson and quantum algebras of variables of Teichmüller spaces induce the **Goldman bracket** on the set of closed geodesics
- *important particular cases*: closing Poisson/quantum algebras on a finite set of **geodesic functions**

$$G_\gamma = 2\cosh(\ell_\gamma/2)$$

- geodesic laminations = sets of non(self)intersecting geodesics; Thurston theory of measured laminations
- modular transformations = graph morphisms modulo symmetries of graphs
- braid group = (sub)group of modular transformations
- total braid group invariants = modular invariants

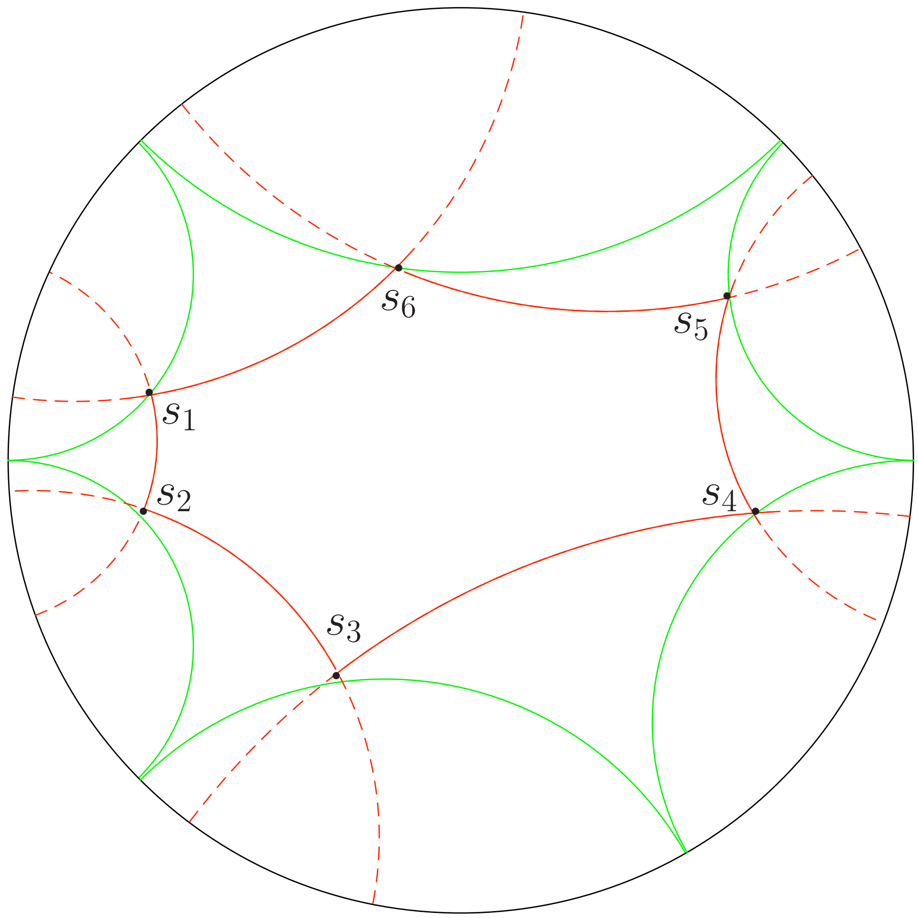
0.1 The group $\mathfrak{G}_n = \Gamma_{0,1,n}$

The simplest example — the Poincaré disc with n marked points s_i, $i = 1, \dots, n$ in the interior. At each point s_i, we introduce the element F_i of the rotation through π; each $F_i = U_i F U_i^{-1}$ is a conjugate of the matrix

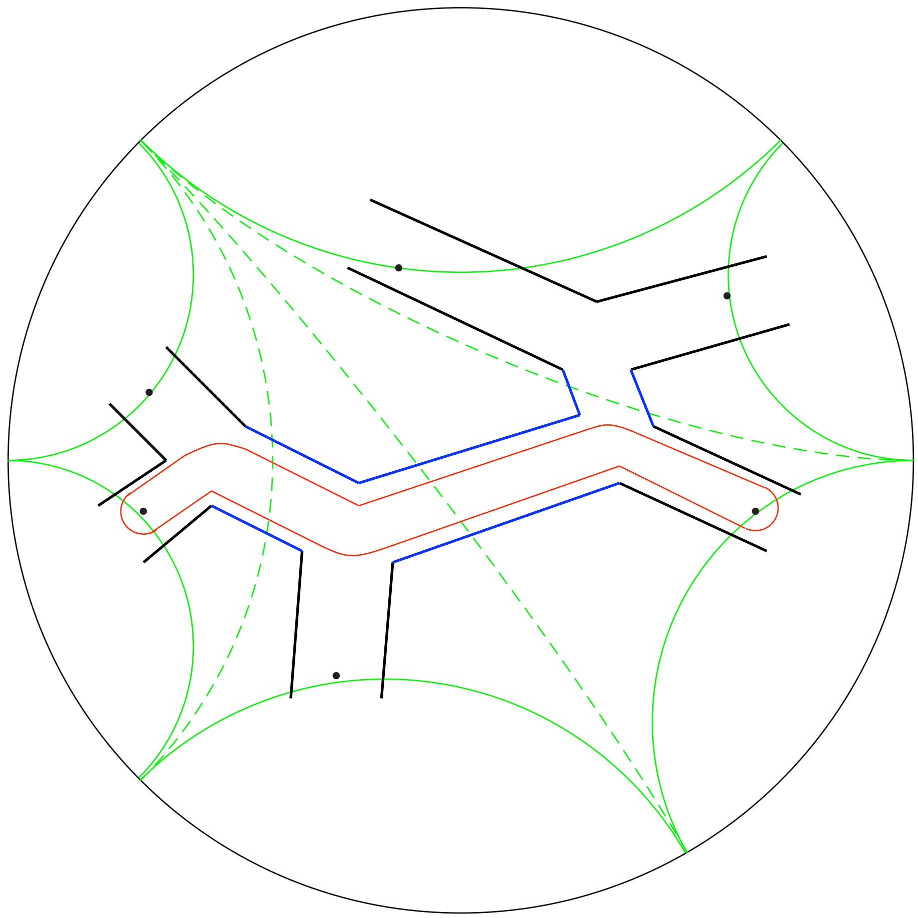
$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The group $\mathfrak{G}_n$ is generated by all the F_i. The element $\gamma_{ij} = F_i F_j$ is, for instance, always a hyperbolic element whose *invariant axis* is a unique geodesic that passes through the points s_i and s_j and its *length* is exactly the double geodesic distance between s_i and s_j (red geodesic lines in the figure).

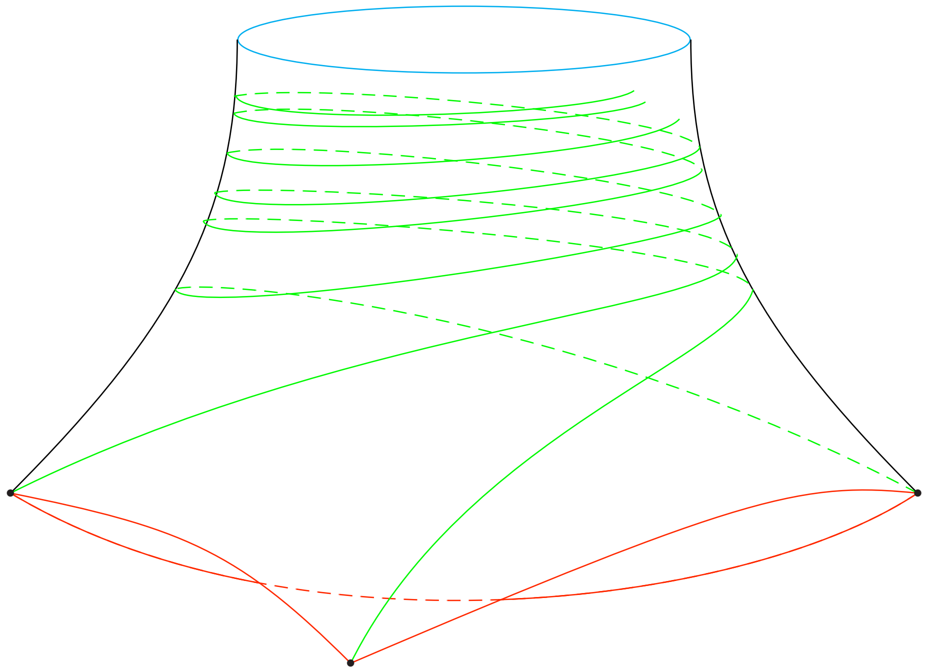
$$G_{ij} = \operatorname{tr} \gamma_{ij}.$$



The Poincaré disc with $n = 6$ orbifold points s_i. The group generated by F_i is Fuchsian (modulo exactly the elements F_i) iff there exists a pattern of green geodesic lines, each passing through exactly one point s_i, that are pairwise parallel at infinity, as shown. We indicate by red geodesic lines the invariant axes of elements $\gamma_{i,i+1} = F_i F_{i+1}$, the part of an axis that lies in the fundamental domain is drawn as the solid line.



The associated fat graph dual to the ideal triangle decomposition of the fundamental domain. Real numbers Z_i, $i = 1, \ldots, 6$ and Y_2, Y_3, and Y_4 associated to all the edges.



The Riemann surface $\Sigma_{0,1,3}$. Lines decomposing into ideal triangles (green) start at the obrifold points and spiral asymptotically to the geodesic boundary of the hole whose perimeter is $\ell_P = |Z_1 + Z_2 + Z_3|$,

Constructing the geodesic functions

1. In each homotopy class of paths there is a unique geodesic.

2. To αth edge correspond a real number Z_α and the Möbius transformation matrix

$$X_{Z_\alpha} = \begin{pmatrix} 0 & -e^{Z_\alpha/2} \\ e^{-Z_\alpha/2} & 0 \end{pmatrix}$$

is inserted each time the path homeomorphic to a geodesic γ passes through the αth edge.

"Right" and "left" turn matrices to be set when a path makes the corresponding turn, and F when we pass around the orbifold point

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(We call the numbers Z_α the coordinates of the corresponding Teichmüller space.)

The geodesic function is then

$$G_\gamma \equiv \operatorname{tr} R X_{Z_{\alpha_n}} L \cdots R X_{Z_{\alpha_3}} R X_{Z_{\alpha_2}} F X_{Z_{\alpha_2}} L X_{Z_{\alpha_1}} =$$

$$= \sum_{j \in J} \exp\left\{ \frac{1}{2} \sum_{\alpha \in E(\Gamma)} m_j(\gamma, \alpha) Z_\alpha \right\},$$

Skein and Poisson structures and mapping class group transformations

The Weil–Petersson bracket B_{WP} is

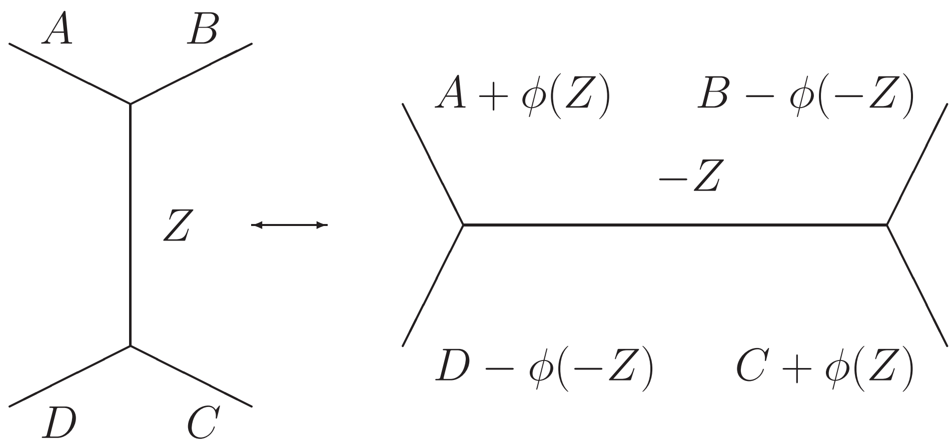
$$B_{\text{WP}} = \sum_{v\in\Gamma}\sum_{i=1}^{3}\frac{\partial}{\partial Z_{v_i}}\wedge\frac{\partial}{\partial Z_{v_{i+1}}},$$

with the summation over **all** three-valent vertices v; v_i, $i = 1,2,3 \bmod 3$ label the cyclically ordered edges incident to this vertex *irrespectively* on whether they are internal or pending edges of the graph.

The center of the Poisson algebra

The center are elements of the form $\sum_{\alpha\in I_j} Z_\alpha$, $j = 1,\dots,s$ with the sum over *all* edges of $\Gamma_{g,s,n}$ in a boundary component of the graph $F(\Gamma_{g,s,n})$ taken with multiplicities. (Each pending edge contributes twice to the corresponding sum).

Whitehead moves on inner edges



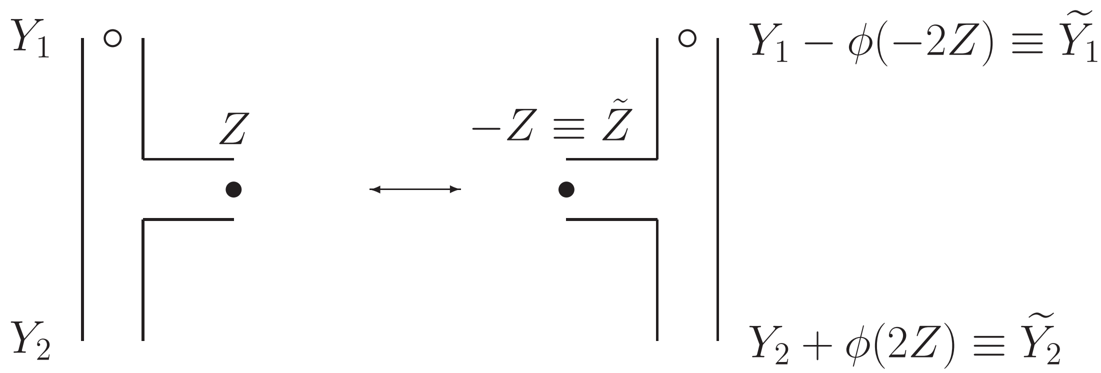
Flip, or Whitehead move on the shear coordinates Z_α. The outer edges can be pending, but the inner edge with respect to which the morphism is performed cannot be a pending edge.

Setting $\phi(Z) = \log(e^Z + 1)$ for shear coordinates of nearby edges, the Whitehead move is a graph morphism

$$W_Z \,:\, (A, B, C, D, Z) \to (A + \phi(Z), B - \phi(-Z), C + \phi(Z), D - \phi(-Z), -Z)$$

Whitehead moves on pending edges

In the case of orbifold surfaces, a new morphism appears.



Flip, or Whitehead move on the shear coordinates when flipping the pending edge Z (indicated by bullet). Any (or both) of edges Y_1 and Y_2 can be pending.

The effect of these transformations corresponds to pushing the "orbifold point" from one boundary component to another or to interchanging two orbifold points in the graph (one or both of edges "Y" can be pending—contain orbifold points).

• open-closed string diagrammatic by Penner and R.Kaufmann.

Action of the mapping class group (the Whitehead moves) on *foliation-shear* coordinates for measured foliations—the pattern of non(self)intersecting geodesics = equal-time lines of string configuration in the *open-closed* pattern.

The tropical limit of transformations for the shear coordinates.

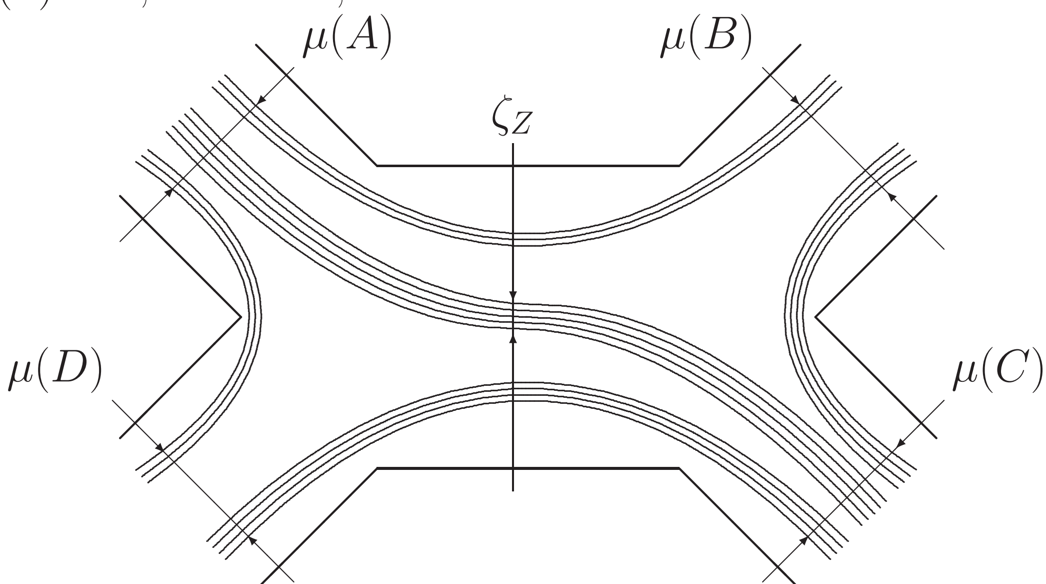
The foliation-shear coordinates of the edges A, B, C, D, and Z are transformed as

$$
\begin{aligned}
M_Z : (\zeta_A, \zeta_B, \zeta_C, \zeta_D, \zeta_Z) &\mapsto \\
\mapsto (\zeta_A + \phi_H(\zeta_Z), \zeta_B - \phi_H(-\zeta_Z), \zeta_C + \phi_H(\zeta_Z),\\
\zeta_D - \phi_H(-\zeta_Z), -\zeta_Z)
\end{aligned}
$$

with

$$\phi_H(\zeta_Z) = (\zeta_Z + |\zeta_Z|)/2,$$

i.e., $\phi_H(x) = x$, for $x > 0$, and zero otherwise.



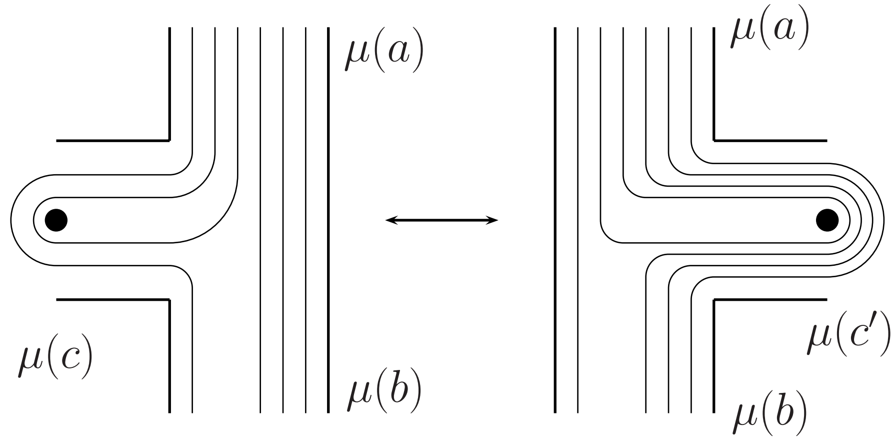
(case $\zeta_Z > 0$)

$$\phi_H(x) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} \phi(\lambda x) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} \phi^{\hbar}(\lambda x),$$

Flipping a pending edge transforms the foliation-shear coordinates of the edges Y_1, Y_2, and Z:

$$M_Z : (\zeta_{Y_1}, \zeta_{Y_2}, \zeta_Z) \mapsto (\zeta_{Y_1} - \phi_H(-2\zeta_Z), \zeta_{Y_2} + \phi_H(2\zeta_Z), -\zeta_Z)$$

with $\phi_H(x)$ above.



$$\mu(c') = 2\max(\mu(a), \mu(b)) - \mu(c)$$

Quantum skein relations

The *quantum* skein relations, which encode both classical skein relation (in the order $O(1)$) and the Poisson algebra (in the order $O(\hbar)$).

For any closed path γ on $\Sigma_{g,s,n}$, define the *quantum geodesic* operator $G^\hbar_\gamma \in \mathcal{T}^\hbar$ to be

$$\begin{aligned} G^\hbar_\gamma &\equiv \,{}^\times_\times \operatorname{tr} RX_{Z^\hbar_{\alpha_n}} L \cdots RX_{Z^\hbar_{\alpha_3}} RX_{Z^\hbar_{\alpha_2}} FX_{Z^\hbar_{\alpha_2}} LX_{Z^\hbar_{\alpha_1}} {}^\times_\times \\ &\equiv \sum_{j\in J} \exp\left\{ \frac{1}{2} \sum_{\alpha\in E(\Gamma_{g,\delta})} \big(m_j(\gamma,\alpha) Z^\hbar_\alpha + 2\pi i\hbar c_j(\gamma,\alpha) \big) \right\}, \end{aligned}$$

where the *quantum ordering* ${}^\times_\times \cdot {}^\times_\times$ implies that we vary the classical expression by introducing additional integer coefficients $c_j(\gamma,\alpha)$, which must be determined from the conditions below.

The defining properties of quantum geodesic functions.

1. If closed paths γ and γ' do not intersect, then the operators $G^\hbar_\gamma$ and $G^\hbar_{\gamma'}$ commute.

2. *Naturality.* The mapping class group $MC(\Sigma_{g,\delta})$ (**??**) acts naturally, i.e., for any $\{G^\hbar_\gamma\}$, $W^\hbar \in MC(\Sigma_{g,\delta})$, and closed path γ in a spine $\Gamma_{g,\delta}$ of $\Sigma_{g,\delta}$, we have

$$W^\hbar(G^\hbar_\gamma) = G^\hbar_{W(\gamma)}.$$

3. *Geodesic algebra.* The product of two quantum geodesics is a linear combination of quantum multicurves governed by the (quantum) skein relation below.

4. *Orientation invariance.* Quantum traces of direct and inverse geodesic operators coincide.

5. *Exponents of geodesics.* A quantum geodesic function $G^\hbar_{n\gamma}$ corresponding to the n-fold concatenation of γ is expressed via $G^\hbar_\gamma$ exactly as in the classical case, namely,

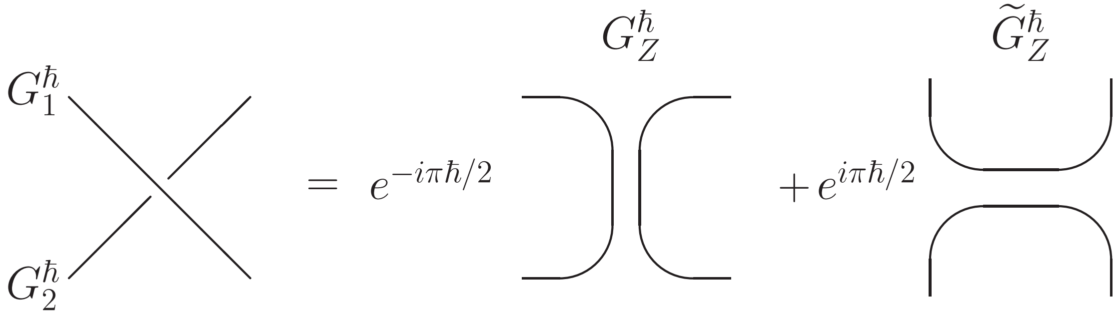
$$G^\hbar_{n\gamma} = 2T_n\big(G^\hbar_\gamma/2\big), \tag{1}$$

where $T_n(x)$ are Chebyshev's polynomials.

6. *Hermiticity.* A quantum geodesic function is a Hermitian operator having by definition a real spectrum.

Let $G_1^\hbar$ and $G_2^\hbar$ be two quantum geodesic operators corresponding to geodesics γ_1 and γ_2. Then

- We must apply the *quantum skein relation*

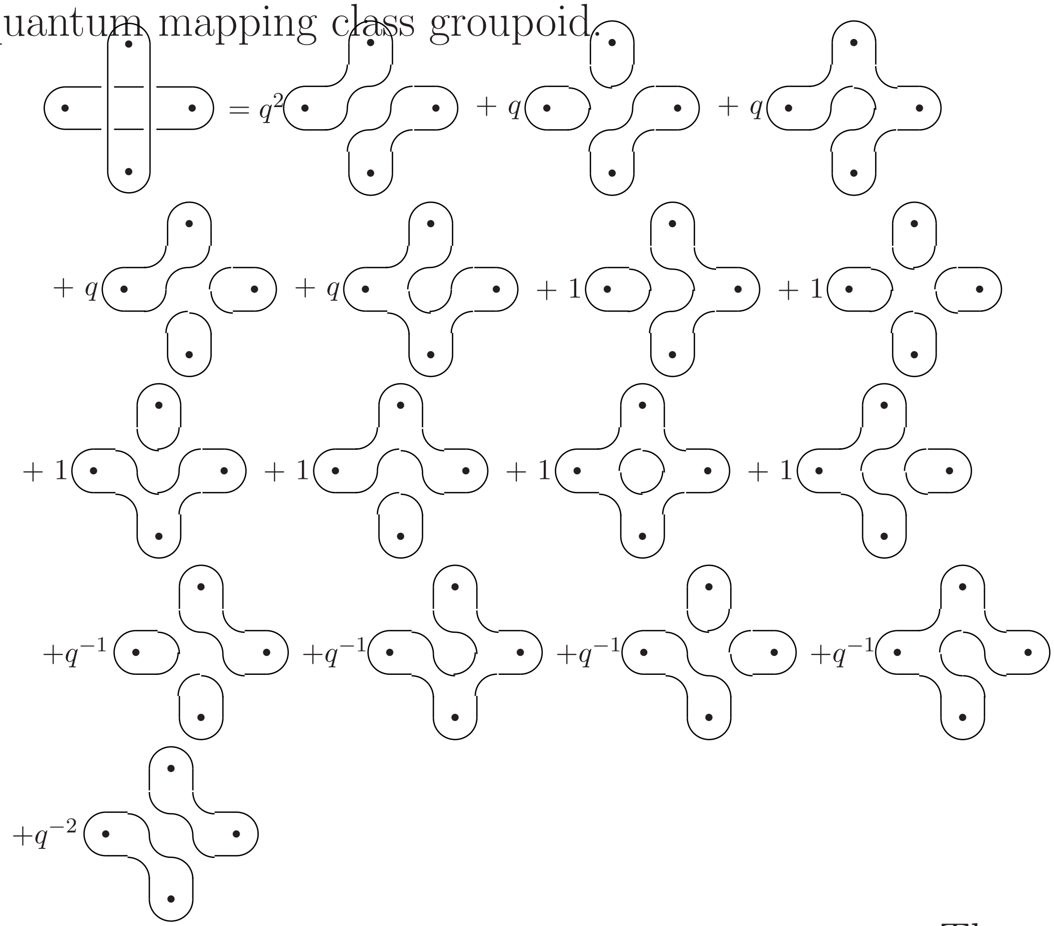


  simultaneously at *all* intersection points.

- After the application of the quantum skein relation we can obtain empty (contractible) loops; we assign the factor $-q-q^{-1}$ to each such loop and this suffices to ensure the quantum Riedemeister moves.

- We can also obtain loops that are homeomorphic to paths encurcling an orbifold point; as in the classical case, the corresponding quantum geodesic functions vanish, so we disregard all such cases of geodesic laminations in the quantum case as well.

There exists a unique quantum ordering ${}^{\times}_{\times}\ldots{}^{\times}_{\times}$ (**??**), which is generated by the quantum geodesic algebra and is consistent with the quantum mapping class groupoid transformations (**??**), i.e., so that the quantum geodesic algebra is invariant under the action of the quantum mapping class groupoid.



The quantum skein relation for the quantum geodesics functions $G^{\hbar}_{ij}$ and $G^{\hbar}_{kl}$ at $i < k < j < l$. Multicurves containing components homeomorphic to passing around the orbifold points vanish. Taking into account the symmetry w.r.t. changing the order of G_{ij} and G_{kl} in the product, we find that only the first (with q^2) and the last (with q^{-2}) terms contribute to the commutator.

Algebras of laminations

$$GL \equiv \prod_{i \in I} G_i^{\hbar},$$

$$GL_1 \cdot GL_2 = \sum_3 q^{s_3} GL_3$$

in the quantum case.

Integrable system: Dubrovin–Ugaglia

We consider the system of equations

$$\frac{d\Psi(\lambda)}{d\lambda} = \left(\sum_{i=1}^{n} \frac{A_i}{\lambda - u_i}\right)\Psi(\lambda),$$

where

$$A_i = E_i\left(-\frac{\mathbb{E}}{2} - V\right),$$

E_i are the projection matrices, s.t. $E_iE_j = \delta_{i,j}E_i$ and $\sum_{i=1}^{n} E_i = \mathbb{E}$, $\mathbb{E}$ is the $n \times n$ unit matrix, V is a skew-symmetric $n \times n$ matrix, and $U = \sum_{i=1}^{n} u_iE_i$. Convenient presentation for matrices in terms of (nonzero) eigenvectors e_i of E_i: $E_i = e_i^T \otimes e_i$, $\langle e_i|e_j\rangle = \delta_{i,j}$.

For two arbitrary solutions $\Psi(\lambda)$ and $\Phi(\lambda)$ of Eq. (0.1), we introduce the bilinear form

$$\langle \Psi(\lambda)(\lambda\mathbb{E} - U)\Phi(\lambda)\rangle .$$

From the defining equations, this form is independent of the point $\lambda \in \mathbb{C}$ if the matrix V is skew-symmetric.

Equivalent matrix (Stokes) system:

$$\frac{d\Psi(z)}{dz} = \left(\frac{V}{z} - U\right)\Psi(z).$$

A convenient choice of the basis of solutions: take $\Psi_j(\lambda)$ such that

$$\Psi_j(\lambda) = \begin{cases} e_j\frac{1}{\sqrt{\lambda-u_j}} + O(\sqrt{\lambda - u_j}), & \lambda \to u_j, \\ g_{ij}e_i\frac{1}{\sqrt{\lambda-u_i}} + O(1)e_i^{\perp}, & \lambda \to u_i, \end{cases}$$

that is, these solutions have no regular part at the point $\lambda = u_j$. Then g_{ij} is just the value of invariant form calculated on the solutions Ψ_i and Ψ_j.

Monodromies

Here are the calculations of algebras of monodromies.

Recall that each $M_i = \mathbb{E} - E_i G$, $G_{ij} = 2g_{ij}$ is the symmetric matrix of monodromy data, $G_{ii} \equiv 2$. Explicitly,

$$M_i = \begin{array}{r} \text{1st} \\ \vdots \\ (i-1)\text{th} \\ i\text{th} \\ (i+1)\text{th} \\ \vdots \\ n\text{th} \end{array} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ -G_{1,i} & \cdots & -G_{i-1,i} & -1 & -G_{i,i+1} & \cdots & -G_{i,n} \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

We have

$$\operatorname{tr} M_i M_j = n - 4 + G_{ij}^2;$$

Note that monodromies around u_i are proportional to *squares* of G_{ij}

Korotkin–Samtleben–Fock–Rosly algebra of monodromies: A_n and D_n

We now adopt Korotkin–Samtleben–Fock–Rosly prescription for the Poisson algebra of the monodromy data:

$$\{M_i, M_j\} = \epsilon(i-j)\left(M_i\Omega M_j + M_j\Omega M_i - M_iM_j\Omega - \Omega M_iM_j\right),$$

where M_i is actually $M_i^{(1)}$ and M_j is $M_j^{(2)}$ – matrices from different spaces; Ω is the R-matrix that permutes these (matrix) spaces entry-by-entry. For $i=j$, we have

$$\{M_i^{(1)}, M_i^{(2)}\} = -M_i^{(1)}\Omega M_i^{(2)} + M_i^{(2)}\Omega M_i^{(1)},$$

where we must indicate explicitly the space index.

Using all the above, we calculate the Poisson bracket

$$\{\operatorname{tr} M_iM_j, \operatorname{tr} M_kM_l\} = \left(\epsilon(i-k) - \epsilon(l-i) - \epsilon(k-j) + \epsilon(l-j)\right)\times$$
$$\times \operatorname{tr}\big([M_i, M_j][M_k, M_l]\big)$$

...in the case of monodromies of Z_2...

$$= \left(\epsilon(i-k) - \epsilon(l-i) - \epsilon(k-j) + \epsilon(l-j)\right)\times$$
$$\times G_{ij}G_{kl}2(G_{il}G_{jk} - G_{ik}G_{jl})$$

in the case of four different indices i, j, k, l (we assume $i<j$ and $k<l$). ϵ-factor takes values -2 when $i<k<j<l$, 2 when $k<i<l<j$, and 0 in all other cases.

The induced bracket on G_{ij}

$$\{G_{ij}, G_{kl}\} = G_{ik}G_{jl} - G_{il}G_{jk} \text{ for } i<k<j<l, \text{ etc.}$$

coincides with the Nelson–Regge bracket on geodesic functions.

Braid group and central elements

Let us construct the upper-triangular matrix $\mathcal{A}$

$$\mathcal{A} = \begin{pmatrix} 1 & G_{1,2} & G_{1,3} & \dots & G_{1,n} \\ 0 & 1 & G_{2,3} & \dots & G_{2,n} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & G_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

associating the entries $G_{i,j}$ with the monodromy data (or with geodesic functions in the geometrical pattern). On the level of monodromies, we have the following transformations:

$$\begin{aligned} &\operatorname{tr} M_k M_l \mapsto \operatorname{tr} M_k M_l, \quad k,l \neq i, i+1, \\ &\operatorname{tr} M_i M_k \mapsto \operatorname{tr} M_{i+1} M_k, \quad k \neq i, \qquad \operatorname{tr} M_i M_{i+1} \mapsto \operatorname{tr} M_i M_{i+1} \\ &\operatorname{tr} M_{i+1} M_k \mapsto \operatorname{tr} M_i M_{i+1} M_k M_{i+1}^{-1}, \quad k \neq i, \end{aligned}$$

and it remains to evaluate the last term. Using that $M_j^2 = \mathbb{E}$ and evaluating the trace, we find that

$$\operatorname{tr} M_i M_{i+1} M_k M_{i+1} = n - 4 + (G_{ik} - G_{i,i+1} G_{i+1,k})^2,$$

which gives the last required term. So, we can present the action of the braid group element $R_{i,i+1}$ exclusively in terms of G_{ij}:

$$R_{i,i+1}\mathcal{A} = \tilde{\mathcal{A}}, \text{ where } \begin{cases} \tilde{G}_{i+1,j} = G_{i,j} & j > i+1, \\ \tilde{G}_{j,i+1} = G_{j,i} & j < i, \\ \tilde{G}_{i,j} = G_{i,j} G_{i,i+1} - G_{i+1,j} & j > i+1, \\ \tilde{G}_{j,i} = G_{j,i} G_{i,i+1} - G_{j,i+1} & j < i, \\ \tilde{G}_{i,i+1} = G_{i,i+1} & \end{cases}.$$

We present this transformation by introducing the special matrices $B_{i,i+1}$ of the block-diagonal form

$$B_{i,i+1} = \begin{array}{c} \\ \\ \vdots \\ i \\ i+1 \\ \vdots \\ \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & G_{i,i+1} & -1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}.$$

Then, the braid group generator $R_{i,i+1}$ acts on $\mathcal{A}$ as

$$R_{i,i+1}\mathcal{A} = B_{i,i+1}\mathcal{A}B^T_{i,i+1} \tag{2}$$

with $B^T_{i,i+1}$ the matrix transposed to $B_{i,i+1}$

Central elements of the Poisson algebra = the braid-group invariants.

The transposed matrix $\mathcal{A}^T$ transforms in the same way:

$$R_{i,i+1}\mathcal{A}^T = B_{i,i+1}\mathcal{A}^T B^T_{i,i+1},$$

and so for any linear combination $\lambda\mathcal{A}+\lambda^{-1}\mathcal{A}^T$, whose determinant is therefore the generating function for the invariants of the braid group:

$$f(\lambda) = \det(\lambda\mathcal{A} + \lambda^{-1}\mathcal{A}^T)$$

(it generates $\left[\frac{n}{2}\right]$ independent central elements = eigenvalues of the monodromy matrix at infinity).

Algebra of geodesic functions

We take $G_{ij} = \operatorname{tr} F_i F_j$ to be the trace of the product of 2×2-matrices of monodromies around the orbifold points, $G_{ij} = -\operatorname{tr} F_i F_j$.

The Poisson relations for all i, j, k, l distinct,

$$\{G_{ij}, G_{kl}\} = (\epsilon(i-k) - \epsilon(l-i) - \epsilon(k-j) + \epsilon(l-j))(G_{il}G_{jk} - G_{ik}G_{jl}),$$

and, in the case of coinciding indices,

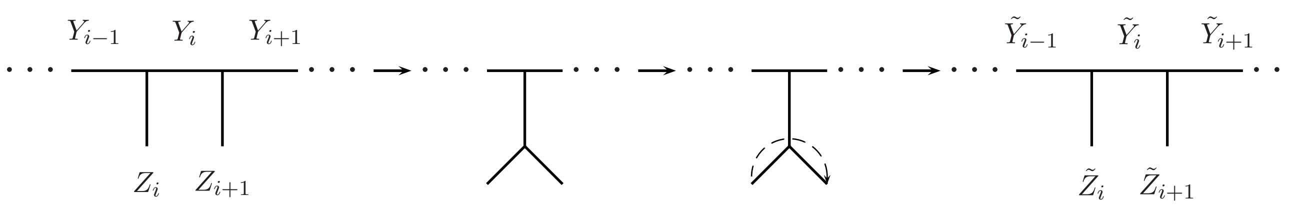
$$\{G_{ij}, G_{kj}\} = (\epsilon(i-k) + \epsilon(k-j) + \epsilon(j-i))(G_{ij}G_{kj} - 2G_{ik}).$$

Actually, *all* possible cases of coinciding indices follow if we set $G_{ii} \equiv 2$ and $\epsilon(0) = 0$.

Poisson relations for geodesic functions exactly coincide with the ones for elements of the monodromy matrices M_i.

Braid group and central elements

The analogue of the braid-group transformation is the three-step sequence of mapping class group transformations:



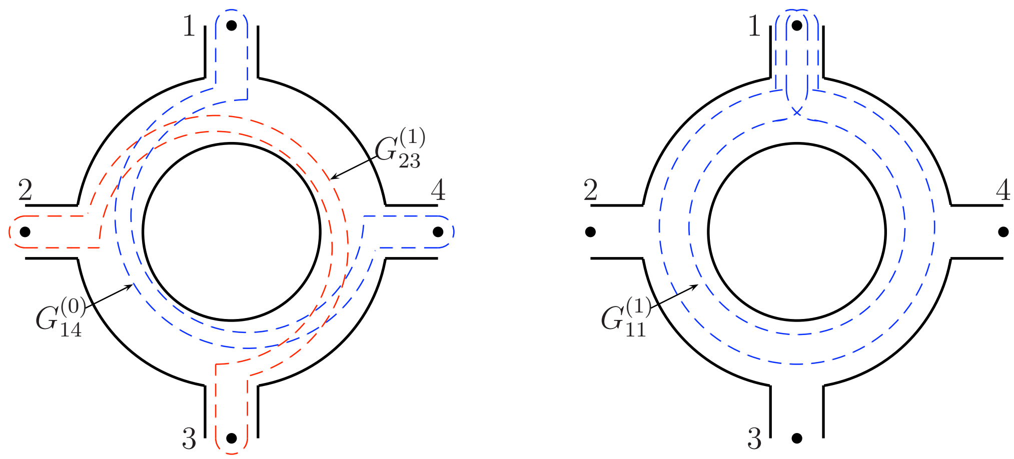


Figure 1: Examples of $G^{(k)}_{ij}$, $i, j = 1, \dots, n$.

Algebras D_n

First, having M_i, $i = 1, \dots, n, n+1, n+2, \dots n+s$, satisfying general KS-FR relations, the system of matrices M_1, M_2, …, M_n, $M_{n+1} \equiv M_5$, where

$$M_5 := M_{n+1} \cdots M_{n+s},$$

satisfies exactly the same KS-FR Poisson algebra.

Elements of D_n

Claim

$$\operatorname{tr} M_i (M_5)^k M_j M_5^{-k} = n + s - 4 + \bigl(G^{(k)}_{ij}\bigr)^2, \qquad G^{(k)}_{ij} \equiv G^{(-k)}_{ji}$$

where each $G^{(k)}_{ij}$, $i, j = 1, \dots, n$ is a polynomial in G_{pq} with $1 \le p < q \le n + s$:

$$G^{(k)}_{ij} = \bigl(G (M_5)^k\bigr)_{ij} = \bigl(G (M_5)^{-k}\bigr)_{ji},$$

where $G = \mathcal{A} + \mathcal{A}^T$ is the symmetric matrix, and the matrix relation

$$G M_5 = M_5^{-T} G$$

holds for any $M_5 = M_{i_1} M_{i_2} \cdots M_{i_s}$.

Algebra of $G_{ij}^{(k)}$:

$$\begin{aligned}
\{G_{ij}^{(q)}, G_{kl}^{(0)}\} \;=\; & \frac{1}{2}\Big[\big(\epsilon(k-j)-\epsilon(l-j)\big)(G_{jl}^{(0)}G_{ik}^{(q)}-G_{jk}^{(0)}G_{il}^{(q)}) \\
& +\big(\epsilon(k-i)-\epsilon(l-i)\big)(G_{il}^{(0)}G_{kj}^{(q)}-G_{ik}^{(0)}G_{lj}^{(q)})\Big]
\end{aligned}$$

$$\begin{aligned}
\{G_{ij}^{(1)}, G_{kl}^{(1)}\} \;=\; & \frac{1}{2}\Big[\big(\epsilon(l-j)+\epsilon(k-i)\big)(G_{il}^{(1)}G_{kj}^{(1)}-G_{ik}^{(0)}G_{lj}^{(0)}) \\
& +\big(\epsilon(j-k)-1\big)(G_{il}^{(2)}G_{kj}^{(0)}-G_{jl}^{(1)}G_{ik}^{(1)}) \\
& -\big(\epsilon(l-i)-1\big)(G_{kj}^{(2)}G_{il}^{(0)}-G_{lj}^{(1)}G_{ki}^{(1)})\Big]
\end{aligned}$$

If unrestricted, the algebra of $G_{ij}^{(k)}$ is infinite (graded).

Small Lemma

$G_{ij}^{(k)}$, $i,j=1,\ldots,n$, commute with G_{pq} for all $n+1\leq p<q\leq n+s$.

Reductions of D_n

We call the level-p reduction

$$M_5^p = \mathbb{E}$$

(likewise all $M_i^2 = 1$)

How to ensure?

It suffices to consider the lower-right part of M_5, the $s \times s$ matrix. Introduce the reduced $s \times s$ matrices

$$\left(\tilde{M}_{n+k}\right)_{ij} = \delta_{ij} - \delta_{ik} G_{n+k,n+j}$$

Then $M_5^p = \mathbb{E}$, if all the roots of the characteristic equation

$$\begin{aligned} &\det(\tilde{M}_{n+1}\tilde{M}_{n+2}\cdots\tilde{M}_{n+s} - \lambda) \\ &= \det(-\tilde{\mathcal{A}}^{-1}\tilde{\mathcal{A}}^T - \lambda) = \det(-\tilde{\mathcal{A}}^T - \lambda\tilde{\mathcal{A}}) = 0 \end{aligned}$$

are pth roots of the unity **not equal to 1**, i.e., $\det(\tilde{\mathcal{A}}^T + \tilde{\mathcal{A}}) \neq 0$.

Claim

Every level-p reduction is Poissonian.

$$G_{ij}^{(k+p)} = G_{ij}^{(k)} (= G_{ji}^{(-k)})$$

$p = 2$, then $G_{ij}^{(0)} = G_{ji}^{(0)}$ and $G_{ij}^{(1)} = G_{ji}^{(-1)} = G_{ji}^{(1)}$, but $G_{ii}^{(1)} \neq 2$.

$p = 3$, then $G_{ij}^{(0)} = G_{ji}^{(0)}$ and $G_{ij}^{(1)} = G_{ij}^{(-2)} = G_{ji}^{(2)}$, so independent elements are $G_{ij}^{(0)}$ with $i < j$ and ALL $G_{ij}^{(1)}$.

$$\# \text{ elements} = \begin{cases} n^2 \times (p/2), \ p \text{ even} \\ n^2 \left(\frac{p-1}{2}\right) + \frac{n(n-1)}{2}, \ p \text{ odd} \end{cases}$$

Action of braid group on D_n

1. Easy part. Action of "old" elements of A_n-type.

If $\mathcal{A}^{(k)}_{ij} \equiv G^{(k)}_{ij}$ whereas $\mathcal{A} \equiv \mathcal{A}^{(0)}$ remains upper triangular, then

$$R_{i,i+1}\mathcal{A}^{(k)} = B_{i,i+1}\mathcal{A}^{(k)}B^T_{i,i+1} \quad \forall k,$$

and the same transformation for $\left(\mathcal{A}^{(k)}\right)^T$.

We need a new element to interchange nth and first indices, $B_{n,1}$

Consider an $\infty\times\infty$ upper triangular matrix composed of $n\times n$ blocks

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathcal{A} & \mathcal{A}^{(1)} & \mathcal{A}^{(2)} & \mathcal{A}^{(3)} & \dots \\ \dots & 0 & \mathcal{A} & \mathcal{A}^{(1)} & \mathcal{A}^{(2)} & \dots \\ \dots & 0 & 0 & \mathcal{A} & \mathcal{A}^{(1)} & \dots \\ \dots & 0 & 0 & 0 & \mathcal{A} & \dots \\ \ddots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Action of $B_{i,i+1}$ with $1\le i<n$ is block-diagonal, and the new element, $B_{n,1}$ has the block-diagonal form interchanging sectors:

$$B_{n,1} = \begin{matrix} \vdots \\ r\cdot n \\ ---- \\ (r\cdot n)+1 \\ \vdots \end{matrix} \left(\begin{array}{cccc|cccc} \ddots & & & & & & & \\ & 1 & & & & & & \\ & & G^{(1)}_{1,n} & & -1 & & & \\ -- & -- & -- & - & - & -- & -- & -- \\ & & 1 & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \end{array}\right).$$

Closure for reduced D_n: $\mathcal{A}^{(k+p)} = \mathcal{A}^{(k)}$.

Consider matrix $\mathbf{A}$ of size $np \times np$:

$$\mathbf{A} = \begin{pmatrix} \mathcal{A} & \mathcal{A}^{(1)} & \mathcal{A}^{(2)} & \dots & \mathcal{A}^{(p-1)} \\ 0 & \mathcal{A} & \mathcal{A}^{(1)} & \dots & \mathcal{A}^{(p-2)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathcal{A} & \mathcal{A}^{(1)} \\ 0 & \dots & 0 & 0 & \mathcal{A} \end{pmatrix}$$

Impossible to close the braid grouprelations for upper triangular form(!)

Way out: Consider $\lambda \mathbf{A} + \lambda^{-1} \mathbf{A}^T$

$$B_{n,1}(\lambda) = \left(\begin{array}{ccc|ccc|ccc} 0 & & & & & & & & \lambda^2 \\ & & & & & & & & \\ & G_{1,n}^{(1)} & & -1 & & & & & \\ \hline & 1 & & 0 & & & & & \\ & & & & & & & & \\ & & & & & G_{1,n}^{(1)} & -1 & & \\ \hline & & & & & 1 & 0 & & \\ & & & & & & & & \\ -\lambda^2 & & & & & & & & G_{1,n}^{(1)} \end{array}\right).$$

The remaining braid group relation is

$$R_{n,1}(\lambda\mathbf{A}+\lambda^{-1}\mathbf{A}^T) = B_{n,1}(\lambda)(\lambda\mathbf{A}+\lambda^{-1}\mathbf{A}^T)B_{n,1}(\lambda^{-1})^T$$

It satisfies the braid group defining relations, for example,

$$R_{n-1,n}R_{n,1}R_{n-1,n} = R_{n,1}R_{n-1,n}R_{n,1} \quad \forall\lambda$$

The central elements are generated by

$$f(\lambda) = \det(\lambda\mathbf{A}+\lambda^{-1}\mathbf{A}^T),$$

their number is $\left[\frac{np}{2}\right]$, the higher Poisson dimension is always even:

$$\text{Poisson leaf dim} = \begin{cases} \frac{n^2p}{2} - \frac{np}{2} = \frac{n(n-1)p}{2},\ p \text{ even, } n \text{ any} \\ \frac{n(n-1)p}{2} - \frac{n}{2},\ p \text{ odd, } n \text{ even} \\ \frac{n(n-1)p}{2} - \frac{n-1}{2},\ p \text{ odd, } n \text{ odd} \end{cases}$$

Perspectives

- hopefully it's just the beginning of a long program
- isomonodromic systems corresponding to D_n and their p-reductions
- relation to geometry and cluster algebras
- corresponding Frobenius manifolds: tomorrow talk by Marta Mazzocco
- quantum version
- ... and many more